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A "MINIMAL"- SET THEORETICAL RESOLUTION OF ZENO'S PARADOX OF "ACHILLES AND TORTOISE"

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Abstract

In this article we analyze Zeno's paradox of "Achilles and Tortoise" using exclusively the theory of infinite sets. In contrast with spatiotemporal based attempts for resolution of the paradox, our interpretation and resolution entails only set theory without making any assumptions on the spatiotemporal structure, since in our opinion Zeno's paradoxes are purely logical paradoxes. This is in accordance to the Eleatic thought, which discarded the "reality" composed by the senses. In particular, we propose a resolution of the paradox from a minimal subset of the set theoretical axioms *ZFC* that is in concord with the mathematics developed in Pythagorean and Eleatic Schools, based on discrete structures.

1. Introduction

Zeno, the Greek philosopher from Elea in southern Italy, who lived from circa 495 to 445 BC, aiming to prove his teacher's Parmenides thesis of monism and falsification of our senses, and hence the impossibility of any spatiotemporal motion, conceived a plethora of

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paradoxical syllogisms that he recorded in a book. Sadly this book has not survived, and what we know of his arguments is second-hand, principally through Aristotle and his commentators. There were apparently 40 ‘paradoxes of plurality’, attempting to show that ontological pluralism–a belief in the existence of many things rather than only one-leads to absurd conclusions. Of these paradoxes only two definitely survive, though a third argument can probably be attributed to Zeno. Aristotle speaks of a further four arguments against motion (and by extension ‘change’ generally), all of which he gives and attempts to refute. In addition, Aristotle attributes two other paradoxes to Zeno. Sadly again, almost none of these paradoxes are quoted in Zeno’s original words by their various commentators, but in paraphrase.

In the book-collective volume “ Zeno’s Paradoxes”, edited by Wesley C. Salmon [16], one could find almost all the major meta-Aristotelian attempts for resolution of Zeno’s paradoxes, in articles of Russell, Bergson, Black, Wisdom, James Thomson, Paul Benacerraf, Owen, as well as the ones of the famous philosopher of science Adolf Grünbaum.

Our proposed resolution adopts the Cantorian interpretation of what an actually infinite set is, and we present an approach based on it. These accomplishments by Cantor are why he (along with Dedekind and Weierstrass) is said by Russell [14] to have “solved Zeno’s Paradoxes.” The argument that this is the correct solution-the so called ‘Standard Solution’-was presented by many people, but it was especially influenced by the elaborated work of Russell [16, lecture 6] and the more detailed work of Adolf Grünbaum [7]. In brief, the argument for the “Standard Solution” is that we have solid grounds for believing our best scientific theories. These scientific theories that present a solid solution to the paradoxes are based on Calculus and its foundational underlying Set Theory, namely, the axiomatic Zermelo-Fraenkel (*ZFC*) including the Axiom of Choice.

In this paper we are in line with the environment of the "Standard Solution" for Zeno Paradoxes of motion, but we adopt only a part of its purely set theoretical branch. That is a system based on a subset of *ZFC* without making any use of Calculus. Furthermore, we shall show in the sequel that our solution is totally new, since we use a much weaker form of *ZFC*, and thus we differ from all the previous attempts. In particular, we address and interpret the Paradox of "Achilles and Tortoise", presented in Aristotle "Physics", as a purely logical paradox. We aim to resolve this paradox assuming only a minimal subset of the axioms of *ZFC* such that it is applicable on discrete structures, avoiding any notions from continuous structures. In such a way, we are in the spirit of Mathematics developed in Pythagorean and Eleatic Schools that were dealing exclusively with discrete structures and countable sets, such as the one of rational numbers $\mathbb{Q}$. We do not even make any reference on continuous structures, not even incommensurable magnitudes that were studied in antiquity, mainly in the works of Theaetetus and Eudoxus, and taught in the later Plato Academy (see Fowler [6], Negrepontis [13]).

In our resolution we adopt the axiomatic system $Z + AC_{\aleph_0}$. That is:

(1) Axiom of extensionality, (2) Null set axiom, (3) Pair set axiom, (4) Union set axiom, (5) Axiom of infinity, (6) Power set axiom, (7) Axiom schema of comprehension (in order to avoid Russell type paradoxes), and in addition we consider, (8) Axiom of countable choice $(AC_{\aleph_0})$.

The set of the above first seven Axioms is referred as the system of Zermelo Axioms (*Z*). It is worth to note that the proposed resolution, based on the above Axiomatic System, holds even for discrete structures and countable sets, and hence is compatible with the mathematics adopted and used by Eleatic and Pythagorean Schools. We emphasize that our interpretation avoids the Axiom of Replacement, the Axiom of Regularity, and the full strength of Axiom of Choice. Rather, we are restricted in the countable version of the Axiom of Choice that it is more intuitively natural than the full version of *AC*. In the sequel, we claim that this Axiomatic System is closer to the notions and techniques developed in the Pythagorean School and are in concord to the spirit of the Eleatic type arguments.

2. The Paradox and its Interpretation

The original statement of the paradox recounted from Aristotle, Physics VI: 9, 239b15 in [2] is the following:

"In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

We assume that in the paradox of 'Achilles and the Tortoise', Achilles, for example, is in a footrace of 100 meters with the Tortoise and that he allows the Tortoise a head start of 10 meters. If we suppose that each racer starts at time t_0, then after some finite time t_1, Achilles will have run 10 meters, bringing him to the Tortoise's starting point. During this time, the Tortoise has run a much shorter distance, say, 1 meter. It will then take Achilles some further time to run that distance, by which time the Tortoise will have advanced farther; and then more time still to reach this third point, while the Tortoise moves ahead. Thus, whenever Achilles reaches somewhere the Tortoise has been, he still has farther to go. Therefore, Achilles must reach where the Tortoise has already been, and since there are an infinite number of points in the race course, the process shall continue ad infinitum. Due to this, Zeno concluded that Achilles can never overtake the Tortoise.

That is exactly where the paradox is created, since, according to our senses, Achilles after some time during the race overcomes the Tortoise, and at t_f he finishes the race, whereas the Tortoise is let say 80 meters from the starting point and definitely before the end of the race.

What is happening spatiotemporally, according to our senses, is given in the following diagram. This situation, according to Zeno’s argument as presented above, is logically impossible. Henceforth, Zeno naturally concludes the falsification of our senses and the impossibility of motion.

In time t_0 : Achilles is at point 0 and Tortoise at point 10.

$$\text{Achilles: } 0 \longrightarrow 10 : \text{Tortoise} \xrightarrow{\hspace{4cm}} \text{End: } 100.$$

In time t_f : Achilles is at the end point of the race, and Tortoise at point 80.

$$0\text{: } \longrightarrow 10 \xrightarrow{\hspace{3cm}} \text{Tortoise}: 80 \xrightarrow{\hspace{2cm}} \text{Achilles-End: } 100$$

Analyzing Zeno’s argument, we observe that Zeno in defending his arguments, and in contrast to the Democretian approach, he adopts the following thesis:

(1) **The infinite divisibility principle**[1]: “*Every existing magnitude can be divided* (*at least logically*) *into an infinite number of nontrivial parts*. *That is, each magnitude is divided to smaller magnitudes and this process continues at infinitum*. *Hence, there is no least magnitude*.”

In addition, as presented above, Zeno assumes that paths have a discrete structure (are comprised by discrete points). Thus, Zeno essentially considers that there is a one to one correspondence between the set of points of Achilles path, and that of Tortoise; since it is not possible for any of them to be at the same position more than one time during the race. Therefore, there exists a one to one correspondence of the points that Achilles passes during the race course, and these of the Tortoise; thesis that is originally presented by Russell [16].

Now we proceed to a further mathematical analysis of the above thesis.

[1]This principle is attributed first to Anaxagoras (see Cohen [3] and Chailos [4]).

Let A be the set of the points-positions of Achilles' path during $t_f - t_0$, that is, $A = [0, 100]$, and let X be the set of the points-positions of Tortoise's path during $t_f - t_0$, that is, $X = [0, 80]$. Then, even though X is a proper subset of $A\,(X \subset A)$, according to our previous analysis for every point $x \in X$, there exists a unique point $a \in A$ that corresponds to $x \in X$, since for every point in Tortoise's path, there exists a corresponding point in the Achilles' path. Mathematically, this is equivalent to the existence of a one to one and onto function $f : A \to X$. Thus, even though the length of Achilles' path is $|A| = 100$ meters and that of Tortoise's path is $|X| = 80$ meters, where X is a proper subset of A, the sets A and X correspond 'one to one' each other. Thus, these sets are infinite of the 'same number' (cardinality in set theoretical language) of elements-points. This phenomenon, of having infinite sets[2] of the same cardinality but of different length/measure, that one is a proper subset of the other, appears paradoxical. And that is exactly where, according to Zeno, the logical paradox appears.

Summarizing the above, our analysis of Zeno's syllogisms that are based on the principle (1) showed that:

(2) *Even though X is a proper subset of A, there is a 'one to one' correspondence between the sets A and X. Hence, A and X contain the 'same number' of (infinite) points-elements.*

In particular, this approach led Zeno to conclude that the statement (2) is logically impossible, and this is because he adopts axiomatically in his arguments the following[3]:

(3) **Whole-parts axiom:** *The size of the 'whole' must be greater than the size of any of its 'proper parts'.*

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[2]Here recall the *Zenonian Principle of Infinite Divisibility* and the Eleatic assumption of the discrete structure of space-time.

[3]This is analyzed extensively in [4].

From this apparent contradiction, Zeno concluded the falsification produced by our senses (the apparent win of the race by Achilles) and henceforth the impossibility of motion (as conceived by our senses).

Strictly speaking (2) and (3) together do not lead to contradiction, and hence the syllogism is an enthymeme, unless one adopts that:

(4) *The number of the elements in any set* (*and in particular in infinite sets*) *measures the size of the set*.

This phenomenon of having infinite sets[4] of the same cardinality but of different length/measure, that in addition one is a proper subset of the other, appeared to Zeno as logically impossible. And that is exactly where according to Zeno the logical paradox appears.

Of course the problem is that the number of elements of a set (loosely speaking its cardinality) does not measure the size/length of the sets[5].

In the next section, where we develop the mathematical framework of our resolution, we prove that this fact is not at all paradoxical, but in contrast that is the exact characterization of infinite sets (indifferently of being discrete or continuum). In this line of thought, we show that even the Axiom of Countable Choice is not required for the resolution of the paradox, but only a weaker form of it. Thus, we can substitute the system $Z + AC_{\aleph_0}$ with $Z + DInf$, where $DInf$ is the statement “*If a set is not Finite, then it is Dedekind Infinite*, *that is*, *there is a* ‘*one to one*’ *correspondence* ‘*onto*’ *one of its proper subsets*”. In relation to this we show among others, assuming the consistency of Z and using results from Model Theory, that the system $Z + DInf$ is strictly weaker than $Z + AC_{\aleph_0}$.

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[4]See note 3.

[5]One could also refer to measure theory where there are infinite sets of Lebesque measure-length zero (see [1]).

3. The Set Theoretical Environment of the Resolution of the Paradox

3.1. The resolution from $Z + AC_{\aleph_0}$

In the previous paragraph, in order to interpret and resolve Zeno's Paradox we adopted the satement *DInf*, that is: "*If a set is not finite* (*standard infinite*), *then there exists a function which is* '*one to one*' *and* '*onto*' *a proper subset of it*." This means that such a set contains a proper subset equipotent to it. Here we mathematically show that this claim, even though initially it appears paradoxical, in contrast, it is a natural consequence of a weak version of the Axiomatic system *ZFC*, namely, the system $Z + AC_{\aleph_0}$ (where $AC_{\aleph_0}$ is a weaker statement of the Axiom of Choice, that is the Axiom of Countable Choice). Finally, we show that the property of a set being *DInf* is even weaker than $AC_{\aleph_0}$ itself.

The axioms of $Z + AC_{\aleph_0}$

Suppose that $\varphi(x_1, x_2, \ldots, x_n)$ is an n-place well-formed formula (wff) in the formal language of set theory $\mathscr{L}$, which is a first order language (as in predicate logic) that in addition to the usual symbol of equality it involves a symbol $\in$, for a binary predicate called *membership*.

(1) **Axiom of extensionality.** If two sets have the same number of elements, then they are identical.

$$\forall x \forall y \forall z[(z \in x \Leftrightarrow z \in y) \Rightarrow x = y].$$

(2) **Null set axiom.** There is an empty set, one which contains no element. $\exists x \forall y \neg(y \in x)$. Using the axiom of extensionality, one can easily see that the empty set is unique. We write $'\varnothing'$ for the empty set.

(3) **Pair set axiom.** If a and b are sets, then there is a set $\{a\}$ whose only element is a and there is a set $\{a, b\}$ whose only elements are a and b.

$$\forall x \forall y \exists z \forall w(w \in z \Leftrightarrow w = x \vee w = y).$$

(4) **Union set axiom.** If a is a set, then there is a set $\bigcup a$, the union of all the elements of a, whose elements are all the elements of a.

$$\forall x \exists y \forall z [z \in y \Leftrightarrow \exists w (w \in x \wedge z \in w)].$$

(5) **Axiom of infinity.** There is a set which has $\varnothing$ as an element and which is such that if a is an element of it, then $\bigcup\{a, \{a\}\}$ (or $a \bigcup \{a\}$) is also an element of it.

$$\exists x [\varnothing \in x \wedge \exists y (y \in x \Rightarrow \exists z (z \in x \wedge \forall w (w \in z \Leftrightarrow w \in y \vee w = y)))].$$

Consequently, this axiom guarantees the existence of a set of the following form:

$$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \cdots\}.$$

(6) **Power set axiom.** For any set x, there exists a set $y = P(x)$, the set of all subsets of x. That is, $\forall x \exists y \forall u [u \in y \Leftrightarrow \forall z (z \in u \Rightarrow z \in x)]$.

(7) **Axiom schema of comprehension.** If a is a set and $\varphi(x, u_1, u_2, \ldots, u_n)$ is a wff in $\mathscr{L}$, where the variable x is free and $u_1, u_2, \ldots, u_n$ are parameters (in a model of the axioms stated up to now), then there exist a set b whose elements are those elements of a that satisfy φ.

$$\forall x \exists y \forall z [z \in y \Leftrightarrow z \in x \wedge \varphi(z, u_1, u_2, \ldots, u_n)].$$

What this essentially means, is that for any set a and any wff H with one free variable and with parameters in a model of set theory (by the axioms defined as of now), there exist a set, unique by the axiom of extensionality, whose elements are precisely those elements of a that satisfy H. We denote this set by $b = \{x \in a : H\}$. Hence, this axiom yields to subsets of a given set. So as a consequence of it, if $f : a \to b$ is a function with domain a, and codomain b, then *Range* (f) is a well-defined set that is a subset of b.

The above seven axioms constitute the system Z (Zermelo).

(8) **Axiom of countable choice:** $AC_{\aleph_0}$. Suppose b is any set and $P \subseteq \mathbb{N} \times b$ any binary relation between natural numbers and members of b, then

$$\forall b[(\forall \in \mathbb{N})(\exists y \in b)P(n, y) \Rightarrow (\exists f : \mathbb{N} \to b)(\forall n \in \mathbb{N})P(n, f(n)].$$

Essentially, the above axiom states that every Countable Family of nonempty sets has a choice function. That is, if $\mathcal{F} = \{A_i\}_{i \in \omega}$ is a countable family of nonempty sets, then there is $f : \mathcal{F} \to \bigcup \mathcal{F}$ such that $f(A_i) \in A_i$ for every $A_i \in \mathcal{F}$.

For the needs of the sequel, we give the formal 'standard' definition of a set to be finite, and a set to be (standard) infinite.

Definition 1 (Finite set)**.** A set X is finite if (and only if) there is a natural number $n \in \mathbb{N}$ such that there is a one to one correspondence between X and n.

Definition 2 (Infinite set (standard))**.** A set X is infinite (standard) if and only if it is not finite (according to Definition 1).

It is of importance to state that the first four axioms are uncontroversial and obvious, in the sense that any set theory within which one wants to be able to do elementary arithmetic would have to contain them. As for the 'Axiom of Infinity', we could naturally conjecture that since the Eleatic School was handling discrete structures and was adopting the notion of infinity-recall *Zenonian Principle of Infinite Divisibility*- it could have accepted this axiom in its arguments. Moreover, since the paradox in study uses the notion of the whole (the set $A = [0, 100]$) given before its parts (all the various segments, distances covered), it is natural to conjecture that the 'Power Set Axiom' is not incompatible with the thought of the Eleatic school. Furthermore, we could naturally argue that Zeno should have accepted the 'Axiom Schema of Comprehension', since in his arguments makes an extensive use of relations on certain domains-sets. We refer to Cohen [3] and Chailos [4], where it is supported that the famous Zeno Paradox "Against plurality" is based on the existence of a certain relation. Now, as for the Axiom of Countable Choice we note the following: In contrast to the full Axiom of Choice (*AC*) that demands the existence of choice functions $f : \mathcal{F} \to \bigcup\mathcal{F}$ for arbitrary families of nonempty sets, the Axiom of Countable Choice ($AC_{\aleph_0}$) justifies only a sequence of independent choices from arbitrary countable families of nonempty sets. In light of the above, $AC_{\aleph_0}$ is more intuitively natural and is in concord with the techniques and the general spirit of mathematics used in the Pythagorean and Eleatic schools, that are based on Discrete/Countable structures. Hence, it could be naturally added to the axioms *Z* for the development of our arguments to lay the ground for a "pure" set theoretical resolution of this paradox of Zeno.

We further study how *DInf* is related to $AC_{\aleph_0}$, and we show that $Z + AC_{\aleph_0} \vdash DInf$.[6] That is, assuming *Z*, the statement *DInf* is a consequence of the Axiom of Countable Choice. Furthermore, we show $Z + DInf \vdash AC_{fin}$, that is, assuming *Z*, the statement AC_{fin}, which denotes the Axiom of Countable Choice on Finite Sets, is a consequence of *DInf*. At the end of the section we show, using results from model theory, that the system $Z + DInf$ is strictly weaker than $Z + AC_{\aleph_0}$, as one could naturally conjecture from what it is stated above. For this, we show that there exists a model $\mathcal{M}$ of *Z*, where *DInf* holds, but $AC_{\aleph_0}$ does not hold. Additionally, it is worth to note that the required statement *DInf* (that every standard Infinite set is Dedekind Infinite) cannot be proved from *Z* or *ZF* alone, since there exists a model of *ZF* (and hence of *Z*) in which there exists a standard Infinite set that is not Dedekind Infinite. Such a model is the Cohen's First Model A4 (see [8]).

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[6] $X \vdash Y$ denotes that the statement *Y* is a logical consequence of *X*. That is, there is a proof of *Y* from *X*.

From now on, when we refer to a set as being Infinite we shall mean infinite in the standard way as in Definition 2.

Statements

(1) Inf : Every infinite set X it is not Finite (This is the standard Infinite set of Definition 2).

(2) $W_{\aleph_0}$: Every infinite set X is Cardinal infinite. That is, it has cardinality at least $\aleph_0$, $\text{card}(X) \geq \aleph_0$.

(3) $CInf$: Every infinite set X is Cantor infinite. That is, it contains a countable subset.

(4) $DInf$: Every infinite set X is Dedekind infinite. That is, it contains a proper subset $B \subset X$, equipotent to it. Hence, $\text{card}(B) = \text{card}(X)$.

In Theorem 1, we show that in $Z + AC_{\aleph_0}$ the definitions (1), (2), (3), and (4) are equivalent, and within the proof of it we emphasize on some fundamental aspects and properties of these statements.

In Theorem 2, we show that in Z, (2), (3), and (4) are equivalent. Thus, the Axiom of Countable Choice is not required for proving the equivalences $W_{\aleph_0} \Leftrightarrow CInf \Leftrightarrow DInf$.

According to the above, (the proof of Theorem 2 follows), we emphasize that if someone defines a set being Infinite according to Dedekind (Dedekind Infinite), then the paradox in concern is resolved only from the (primitive) Axiomatic Set Theory Z. It is important to note that the *Zenonian Principle of Infinite Divisibility*, that is clearly assumed by Zeno as we presented in Section 1, points to what we nowadays call a countable (and 'dense in themselves') infinite set (see [11]). Such sets (e.g., the set of rational numbers $\mathbb{Q}$) have the property of being Dedekind Infinite.

Indeed, if X is a countable set (that is there is a bijection form X to ω), then $X = \{a_i\}_{i=0}^{\omega}$ (where ω is the first nonzero limit ordinal of cardinality $\aleph_0$). Take now the set $Y = X \setminus \{a_0\}$ which is a proper subset of X with cardinality $\aleph_0$. To see that X is Dedekind Infinite consider the function $h : X \to Y \subset X$, where $h(a_i) = a_{i+1}$ for every $i \in \omega$. Clearly h is 'one to one', since if $a_i \neq a_j$, $h(a_i) = a_{i+1} \neq a_{j+1} = h(a_j)$, and furthermore h is also 'onto', since $Range(h) = Y$. Thus, $h : X \to Y \subset X$ is a bijection from X onto a proper subset of it, and thus X is a Dedekind Infinite set.

Theorem 1. *In* $Z + AC_{\aleph_0}$ *the statements* (1), (2), (3), *and* (4) *are equivalent*. *That is*, *a set is infinite if and only it is not finite*, *if and only if* $card(X) \geq \aleph_0$, *if and only if it is Cantor Infinite*, *if and only if it is Dedekind Infinite*.

Proof. For this, we show in (a)-(c) that $Inf \Rightarrow CInf \Rightarrow DInf \Rightarrow Inf$ and in (d) that $CInf \Leftrightarrow W_{\aleph_0}$.

(a): (1) implies (3). That is, $Inf \Rightarrow CInf$ (every infinite set contains a countable subset). The proof is standard if we assume the Axiom of Replacement and the Axiom of Choice in its full strength (see Appendix Lemma A1). Here we prove it only from $Z + AC_{\aleph_0}$.

Let X be an infinite set. For every $n \in \mathbb{N}$, let A_n be the set of all subsets of X with cardinality 2^n. Thus, the members of A_n are sets of cardinality 2^n. Since X is an infinite set, $\forall n \in \mathbb{N}$, $A_n \neq \varnothing$. Apply $AC_{\aleph_0}$ to the family $\mathcal{F} = \{A_i\}_{i \in \mathbb{N}}$ in order to obtain a function $\phi : \mathcal{F} \to \bigcup \mathcal{F}$ and via it a subsequence $\{B_n\}_{n \in \mathbb{N}} = \phi[\mathcal{F}]$, where each B_n is a subset of X of cardinality 2^n. (It is worth to note that the Axiom of Replacement is not required since $\{B_n\}_{n \in \mathbb{N}} = \phi[\mathcal{F}]$ is the image in a predefined set of a well-defined function on a set.) Observe that the sets $\{B_n\}_{n\in\mathbb{N}}$ are also distinct. We define recursively the sequence $\{C_n\}_{n\in\mathbb{N}}$ of mutually disjoint sets as follows (this is a standard technique in analysis):

$$C_0 = B_0,\ C_1 = B_1 \setminus C_0,\ \dots,\ C_n = B_n \setminus \bigcup_{i=1}^{n-1} C_i,\ \dots.$$

Obviously, $1 \le \mathrm{card}(C_n) \le 2^n (\text{since} \sum_{i=0}^{n-1} 2^i = 2^n - 1)$. Now, it is trivial to observe that the family $\mathcal{S}' = \{C_n\}_{n\in\mathbb{N}}$ is a family of nonempty and mutually disjoint sets. A second application of $AC_{\aleph_0}$ to the nonempty family $\mathcal{S}' = \{C_n\}_{n\in\mathbb{N}}$ guarantees the existence of a choice function $\psi : \mathcal{S}' \to \bigcup\mathcal{S}'$, and through this we obtain a set $C = \{c_n\}_{n\in\mathbb{N}} = \psi[\mathcal{S}']$, such that $c_i \neq c_j$, $\forall i \neq j \in \mathbb{N}$. This set $C = \{c_n\}_{n\in\mathbb{N}}$ is the countable set that establishes X to be Cantor Infinite. (As done earlier in the proof, the Axiom of Replacement is not required since $C = \{c_n\}_{n\in\mathbb{N}} = \psi[\mathcal{S}'].$) □

(b): (3) implies (4). That is, $CInf \Rightarrow DInf$ (if a set is Cantor Infinite then it is Dedekind Infinite). Since X is Cantor Infinite, it contains a countable subset $B \subseteq X$. That is, there exists a bijection $f : \mathbb{N} \mapsto B$. Define the function $h : X \mapsto X$ as follows: $h(x) = f(n+1)$ if $x = f(n)$, $x \in B$ and $h(x) = x$, if $x \in X \setminus B$. (Thus, the action of h on B is identified with the action of f on the successor of each element of $\mathbb{N}$, that leaves the remaining elements of X invariant.)

Claim 1. The function $h : X \mapsto X$ is 'one to one'.

Proof of Claim 1. Let $x, y \in X$ with $h(x) = h(y)$. In order for such a thing to hold we can easily see that either both $x, y \in B$, or $x, y \in X \setminus B$.

In the first case, where $x, y \in B$, we have $x = f(n)$, $y = f(m)$ for some $n, m \in \mathbb{N}$. From the definition of $h, h(x) = h(y) \Rightarrow f(n+1) = f(m+1)$, and since f is 'one to one', $n + 1 = m + 1$ and hence $n = m$. Thus, $x = y$.

In the second case, where $x, y \in X \setminus B$, we conclude that $h(x) = h(y) \Rightarrow x = y$. This proves Claim 1.

Let $D = h(X) \subseteq X$. Then the function $h : X \mapsto D$ from Claim 1 is ‘one to one’ and since $D = Range\ h$, then $h : X \mapsto D$ is also onto and hence a bijection. Thus, X is equipotent to its subset $D = h(X)$. To complete the proof, it is enough to show the following.

Claim 2. The set $D = h(X)$ is a proper subset of X.

Proof of Claim 2. Let $x = f(0)$. We claim that $x \notin D$. Indeed, if we suppose that $x \in D$, then $x = h(z)$ for some $z \in X$, and we have the following two cases:

(i) If $z \in B$, then $z = f(n)$ for some $n \in \mathbb{N}$. Hence by the definition of $x = f(0) = h(z) = f(n+1)$. Thus, $f(0) = f(n+1)$. Since f is ‘one to one’, $0 = n + 1$. This leads to a contradiction since $0 \in \mathbb{N}$ is not a successor.

(ii) If $z \in X \setminus B$, then $x = f(0) = h(z) = z$ and thus (from the definition of $f : \mathbb{N} \mapsto B$) $z \in B$. This leads again to contradiction (since by hypothesis $z \in X \setminus B$).

Putting everything together, we conclude that there exists $x \in X \setminus h(X)$ and therefore $D = h(X)$ it is a proper subset of X. This proves Claim 2. □

The proof of part (b) is now complete. □

Remark 1. Observe that in (b) of the above proof we have shown $CInf \Rightarrow DInf$, without using the Axiom of Countable Choice.

(c): (4) implies (1). That is, $DInf \Rightarrow Inf$. Suppose, for contradiction, that X is a finite, yet a Dedekind infinite set. It is well known that in Z (and hence in $Z + AC_{\aleph_0}$) the Pigeonhole principle holds: “If a set is finite, then it does not contain a proper subset equipotent to it.” (see the appendix for details). Hence, X does not contain a proper subset equipotent to it. This contradicts the hypothesis of X being Dedekind Infinite (and thus containing a proper subset equipotent to it). Therefore, (4) implies (1). ☐

Remark 2. Observe that in (c) we have shown $DInf \Rightarrow Inf$, without using the Axiom of Countable Choice.

(d): (2) and (3) are equivalent: That is, $CInf \Leftrightarrow W_{\aleph_0}$.

$CInf \Rightarrow W_{\aleph_0}$. Suppose that X is Cantor Infinite set. Then X contains a countable subset and hence there is a one to one map from $\mathbb{N}$ into X. Thus, by the definition of cardinality of a set, $\aleph_0 \leq card(X)$. ☐

$W_{\aleph_0} \Rightarrow CInf$. Suppose that X is a set such that $card(X) \geq \aleph_0$. Hence there is an injection $g : \mathbb{N} \mapsto X$. Thus, $g[\mathbb{N}] \subseteq X$ is a countable set and hence X is Cantor Infinite. ☐

Remark 3. Observe that in (d) we have shown $CInf \Leftrightarrow W_{\aleph_0}$, without using the Axiom of Countable Choice.

Putting (a), (b), (c), (d) together we conclude that in $Z + AC_{\aleph_0}$.

$Inf \Leftrightarrow W_{\aleph_0} \Leftrightarrow CInf \Leftrightarrow DInf$. The proof of the theorem is now complete. ☐

Theorem 2. *In the axiomatic system Z the statements* (2), (3), *and* (4) *are equivalent. That is,* $W_{\aleph_0} \Leftrightarrow CInf \Leftrightarrow DInf$.

Proof. From Remark 3 $CInf \Leftrightarrow W_{\aleph_0}$ in Z, and from Remark 1 $CInf \Rightarrow DInf$ in Z. It remains to show that in Z, $DInf \Rightarrow CInf$. That is, if X is Dedekind Infinite, then X it is also Cantor Infinite. To this end, since X is Dedekind infinite, it contains a proper subset B equipotent to it. Thus, $\exists f : X \to B \subset X$ which is 'one to one' and 'onto' $B \subset X$.

Consider $a \in X \setminus B$. Then, $a \notin Range(f)$. To show that X is Cantor Infinite we construct recursively a well-defined Infinite countable subset Y of X as follows:

Consider $f : X \times X$ as above, then by recursion theorem (see 5.6 in [12]) there is a unique function $g : \mathbb{N} \to X$, such that:

$$g(0) = a \text{ and } g(n+1) = f((g(n)) \text{ for all } n \in \mathbb{N}.$$

The set $Y = Range(g) = \{g(n)\}_{n\in\mathbb{N}}$ is a well-defined countable subset $Y \subseteq X$. Indeed, by construction, for any $n \in \mathbb{N}$, $g(n) = f^{(n)}(a)$ where $f^{(n)}$ denotes the n^{th} recursive iteration of the action of f on a. If (w.l.o.g) $\exists i < j$ such that $g(i) = g(j)$, then $f^{(i)}(a) = f^{(j)}(a)$ and hence $f^{(j-i)}(a) = a$ where $j - i \neq 0$. Thus, $a \in Range(f)$, and this contradicts the choice of a. Hence, g is a 'one to one' well-defined function. This set Y is the Infinite countable set we are looking for.

3.2. $Z + DInf$ **is weaker than** $Z + AC_{\aleph_0}$

In this section we show that the statement *DInf*, that is, "every (standard) Infinite set is Dedekind Infinite" which is, as we have shown, fundamental in our resolution of the paradox, it is not at all paradoxical and it is even weaker from the intuitively valid *Axiom of Countable Choice*. Henceforth, it could be naturally added to the axioms *Z* (instead of $AC_{\aleph_0}$) in order to develop our thesis. At first, we prove some interesting results for the relation between *DInf* and $AC_{\aleph_0}$.

Theorem 3. $Z + DInf \vdash AC_{fin}$.

That is, assuming *Z* and the statement *DInf* (that is every (standard) Infinite set is Dedekind Infinite), the Axiom of Countable Choice on Finite Sets (AC_{fin}) holds.

Proof. Let $\mathcal{F} = \{X_n\}_{n\in\mathbb{N}}$ be a countable family of nonempty finite sets. We show that $\prod_{n\in\mathbb{N}} X_n \neq \varnothing$ and hence AC_{fin} holds. For this, define $X = \bigcup_{n\in\mathbb{N}} (X_n \times \{n\})$. Then X is not a finite set and thus by hypothesis it is a Dedekind Infinite Set. Thus, by Theorem 2, the set X is Cantor infinite and hence there is an injection $f : \mathbb{N} \mapsto X$. Since $\forall n \in \mathbb{N}$, X_n is a finite set, using proof by contradiction, we can easily show that $M = \{n \in \mathbb{N} : f[\mathbb{N}] \cap (X_n \times \{n\}) \neq \varnothing\}$ is a (standard) infinite set.

Claim 1. $\prod_{n\in M} X_n \neq \varnothing$.

Proof of Claim 1. Since $f[\mathbb{N}] \cap (X_m \times \{m\}) \neq \varnothing$, for every $m \in M$ define $n(m) = \{n \in \mathbb{N} : f(n) \in X_m \times \{m\}\}$. Thus, for every $m \in M$, there exists a unique $x_m \in X_m$ such that $f(n(m)) = (x_m, m)$. Therefore, there exists an element $(x_m)_{m\in M} \in \prod_{m\in M} X_m$, thus $\prod_{n\in M} X_n \neq \varnothing$. □

According to the above, to complete the proof, it is enough to show the following claim.

Claim 2. If $\mathcal{F} = \{X_n\}_{n\in\mathbb{N}}$ is a countable collection of nonempty finite sets such that it exists an infinite set $M \subseteq \mathbb{N}$ with the property $\prod_{m\in M} X_m \neq \varnothing$, then $\prod_{n\in\mathbb{N}} X_n \neq \varnothing$. (Hence, the Axiom of Countable Choice for finite sets holds.)

Proof of Claim 2. Consider the sequence of nonempty finite sets $\langle X_n \rangle_{n\in\mathbb{Z}_+}$ and define $Y_k = \prod_{n\leq k} X_n$. Inductively, observe that for every $k \in \mathbb{Z}_+$, the sets Y_k are nonempty and are finite. Now consider the sequence of nonempty finite sets $\langle Y_k \rangle_{k\in\mathbb{Z}_+}$. By hypothesis, there exists a finite set $M \subseteq \mathbb{Z}_+$ with the property $\prod_{m \in M} Y_m \neq \varnothing$, and hence, there exists an element $(y_m)_{m \in M} \in \prod_{m \in M} Y_m$ with $y_m = (x_1^m, x_2^m, \ldots, x_m^m) \in \prod_{n \leq m} X_n$. Equivalently, $\forall 1 \leq i \leq m \in M$, $x_i^m \in X_i$. $(^*)$

Now consider the set $Z_n = \{m \in M : n \leq m\}$. Since $M \subseteq \mathbb{Z}_+$ is an infinite set, we get that $\forall n \in \mathbb{Z}_+ \, \exists m \in M : n \leq m$, and hence $\forall n \in \mathbb{Z}_+$ the set Z_n being a nonempty subset of $\mathbb{N}$ has a minimum element. Define $m(n) = \min Z_n = \min\{m \in M : n \leq m\}$. Since, $y_{m(n)} = (x_1^{m(n)}, x_2^{m(n)}, \ldots, x_{m(n)}^{m(n)}) \in \prod_{n \leq m(n)} X_n$ and $n \leq m(n)$, then form $(^*)$, $\forall n \in \mathbb{Z}_+, x_n^{m(n)} \in X$. The sequence $f : \mathbb{Z}_+ \mapsto \bigcup_{i \in \mathbb{Z}_+} X_i$, with $f(n) = x_n^{m(n)} \in X_n$, is a choice function for the family $\mathcal{F} = \{X_n\}_{n \in \mathbb{Z}_+}$ with $(x_n^{m(n)})_{n \in \mathbb{Z}_+} \in \prod_{n \in \mathbb{Z}_+} X_n$. Therefore $\prod_{n \in \mathbb{Z}_+} X_n \neq \varnothing$, and since $X_0 \neq \varnothing$, then $\prod_{n \in \mathbb{N}} X_n \neq \varnothing$. This finishes the proof of Claim 2.

The proof of Theorem 3 is now complete. □

In what follows, we show, using results from model theory, that the statement *DInf* is indeed strictly weaker than the Axiom of Countable Choice (assuming consistency of *Z*). For this, we show that there exists a model $\mathcal{M}$ of *Z* in which every (standard) Infinite set is Dedekind Infinite, yet the Axiom of Countable Choice fails.

Suppose that k is an aleph. The statement W_k states that for every set *X*, either $card(X) \leq k$ or $card(X) \geq k$.

For the sequel we also need the notion of *cofinality* of an ordinal number.

Definition 3 (Cofinality)**.** If α is an ordinal number, then the cofinality of α, $cf\ (\alpha)$, is the least ordinal number ϑ such that α is the limit of an increasing sequence of ordinals of length ϑ.

Note that if α is a successor ordinal, then $cf(\alpha) = 1$ and $cf(\aleph_\alpha) = \aleph_\alpha$ (Theorem 9.2.4 of [9]). Moreover $cf(\omega) = \omega$, since ω is the first countable limit ordinal. For limit ordinals the following it is also true:

Lemma 1. *If* α *is a limit ordinal, then* $cf(\aleph_\alpha) = cf(\alpha)$.

Proof of Lemma 1. Since α is a limit ordinal, by definition $\alpha = \sup\{\beta : \beta < \alpha\}$. Using transfinite recursion we construct an increasing subsequence $\langle \beta_\nu \rangle_{\nu < cf(\alpha)}$ of least length $cf(\alpha)$ that has limit α; that is, $\alpha = \lim_{\nu \to cf(\alpha)} \beta_\nu$. Moreover, $\aleph_\alpha = \bigcup_{\beta < \alpha} \aleph_\beta$. Since $f : \alpha \to \aleph_\alpha$, where $f(\gamma) = \aleph_\gamma$ for any $\gamma < \alpha$ is an increasing bijection, we conclude that $\langle \aleph_{\beta_\nu} \rangle_{\nu < cf(\alpha)}$ is an increasing subsequence of $\langle \aleph_\beta \rangle_{\beta < \alpha}$ such that $\aleph_\alpha = \lim_{\nu \to cf(\alpha)} \aleph_{\beta_\nu}$. Form the definition of cofinality of α, this sequence is of least length $cf(\alpha)$. Hence, $cf(\aleph_\alpha) = cf(\alpha)$. □

The next theorem is Theorem 8.6 of [10].

Theorem 4. *Let* $\aleph_\alpha$ *be a singular aleph. Then there exists a model of ZF in which*

(i) *For each* $\lambda < \aleph_\alpha$, W_λ *holds.*

(ii) $W_{\aleph_\alpha}$ *fails and* $AC_{cf(\aleph_\alpha)}$ *fails.*

Lemma 2. *In Z the statement DInf is strictly weaker than* $AC_{\aleph_0}$.

Proof. From the proof of Theorem 1, $Z + AC_{\aleph_0} \vdash DInf$. Thus, any model of Z that satisfies $AC_{\aleph_0}$, it also satisfies $DInf$. Hence, in order to prove that the system $Z + DInf$ is weaker than $Z + AC_{\aleph_0}$, it is enough to show that assuming the consistency of Z, there exists a model $\mathcal{M}$ of Z that satisfies $DInf$ in which $AC_{\aleph_0}$ fails. Since ω is a limit ordinal, by Lemma 1 $cf(\aleph_\omega) = cf(\omega) = \omega$. We set $\aleph_\alpha = \aleph_\omega$ and $\lambda = \aleph_0 < \aleph_\omega$ in Theorem 4 to conclude that there is a model $\mathcal{M}$ of ZF (and hence of the weaker Z) in which $W_{\aleph_0}$ holds, but on the other hand $AC_{cf(\omega)} \equiv AC_\omega \equiv AC_{\aleph_0}$ fails. Since from Theorem 2 the statements $W_{\aleph_0}$ and $DInf$ are equivalent in Z, then in $\mathcal{M}$ the statement $DInf$ holds, but the statement $AC_{\aleph_0}$ fails. That is what we wanted to prove. □

4. Comments and Conclusion

One could argue that the argument derived from Zeno's paradoxes against taking a continuum to be made out of points is circumvented, because the "construction" of real numbers, within which we think about a point in the continuum, is not a geometrical construction. It is not a matter of distributing points in space, but of defining the real numbers and showing that these can be ordered so as to be order isomorphic to the points on a line (see p. 93 of [15]). Of course in order to do this, we have to use an equivalent version of the Axiom of Choice (*AC*), in its full strength version, namely the Well Ordering Principle (*WOP*) (see 8.1.13 in [10]). In this setting, since the set of real numbers $\mathbb{R}$ (by its construction) is a complete ordered field, then every nonempty subset of it has a supremum and thus, the concepts of convergence and infinite sums required by the standard interpretation of Zeno's paradoxes are formalized. Of course such knowledge was not existed in Zeno's era, and the closest analogue to this was the later definition of the criterion of the ratio of incommensurable magnitudes according to Eudoxus (404-355 B.C.).

Eudoxus' concept is the analogue of what we call now Dedekind cuts, which by definition constitute the set-structure of $\mathbb{R}$ (see Section 4 of [5] and [6]).

In respond to the above criticism, one could observe that our analysis of Zeno's arguments, as well as our proposed resolution, differ from the "standard" one. It does not depend on modern calculus that requires the axiom of continuity and the well ordering of the set of real numbers $\mathbb{R}$, as well as the concepts of supremum, convergence, and infinite series. Furthermore, our resolution neither requires the notion of completeness of $\mathbb{R}$, nor the distinction between countable and uncountable infinities. In this way our foundational approach is much closer to the one adopted by the Pythagorean and Eleatic Schools of thought, which is based on discrete structures and on countable sets, as that of the rational numbers $\mathbb{Q}$. In light of the above, our resolution is valid even in the case that we deal solely with countably infinite sets. It is based only on the set theoretical fact that the set of points in the distance-interval covered by Achilles, as well as the corresponding one covered by Tortoise are both infinite sets of finite length (Lebesque measure). In this context, we have shown that the existence of a one to one function from an infinite set onto a proper subset of it is not at all paradoxical, but in contrast, this property is equivalent (under certain hypotheses) to the defining property of a set to be Infinite (actually Dedekind Infinite). For establishing and formalizing our arguments we presented formal set theoretical proofs based on the Axiomatic System $Z + AC_{\aleph_0}$, which is strictly weaker than *ZFC*. Furthermore, we showed that our arguments can be even derived from the strictly weaker system $Z + DInf$.

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Appendix

In this Appendix we prove auxiliary results mentioned in the paper. At first we provide a sketch of another proof of the fundamental result $Inf \Rightarrow CInf$ of Theorem 2. This proof assumes the axiomatic system ZFC and uses explicitly the Axiom of Replacement and the full version of the Axiom of Choice (Chapter 8, Theorem 1.4 of [9]).

Lemma A1. *If A is a (standard) infinite set, then A contains a countable subset.*

Sketch of proof (*ZFC*). Let A be an infinite set. From the Axiom of Choice (AC), equivalently from the Well Ordering Theorem (WOT), A is 'well orderable set', and hence using the Axiom of Replacement it is isomorphic with a unique ordinal number Ω (see 6.3.1 in [9]). Now using the principle of Transfinite Recursion and the Axiom of Replacement (once more) we conclude that the set A can be well ordered in a transfinite sequence of length Ω, thus, $A = \langle a_\alpha : \alpha < \Omega \rangle$. From the defining properties of ordinals and well ordered sets, $C = \langle a_\alpha : \alpha < \omega \rangle$, where ω is the first countable limit ordinal, is an initial segment of A. Hence, the $Range(C) = \{a_\alpha : \alpha < \omega\}$ is a countable subset of A.

In the next lemma we sketch a proof of 'Pigeonhole Principle' that is used in proving $DInf \Rightarrow Inf$ in Theorem 1.

Lemma A2 (Pigeonhole principle)**.** *If A is a finite set, then there is no "one to one function" $f : A \to A$ onto a proper subset $B \subset A$. (Equivalently, if A is a finite set, then every 'one to one' function $f : A \to A$ is also 'onto' A, and hence it is a bijection.)*

Sketch of Proof. We claim that to prove Pigeonhole principle it is enough to show the following:

Claim. If $m \in \mathbb{N}$ and if $g : [0, m) \mapsto [0, m)$ is any 'one to one' function, then it is also an 'onto' function. (Recall that in the construction of $\mathbb{N}$, $[0, m)$ is a subset of $\mathbb{N}$ that is identified with m.)

Indeed, assuming the claim, consider any 'one to one' function $f : A \mapsto A$. Set $\pi : A \mapsto [0, m)$ the bijection that witnesses the finiteness of A for some $m \in \mathbb{N}$. Such a bijection exists since $card(A) = m$ for some $m \in \mathbb{N}$. Then the function $g = \pi \circ f \circ \pi^{-1}$ is well defined and it is 'one to one'. Hence, by the claim it is onto $[0, m)$. Thus,

$f = \pi^{-1} \circ g \circ \pi$ is a bijection, as a composition of bijections, and hence it is onto A. Now the proof of the mentioned claim is a well-known and standard application of the Induction Principle (see 5.27 of [12]). (Alternatively, it is an immediate from the fact that any element of $\mathbb{N}$ is a finite cardinal number (see Chapter 5.2 of [9]).)

□