ON A CLASS OF HOLOMORPHIC FUNCTIONS REPRESENTABLE BY CARLEMAN FORMULAS IN SOME CLASS OF BOUNDED, SIMPLY CONNECTED DOMAINS FROM THEIR VALUES ON AN ANALYTIC ARC.

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Abstract. Let $\mathcal{U}$ be a bounded, simply connected domain with Jordan rectifiable boundary and let $M \subset \partial\mathcal{U}$ be an open analytic arc whose Lebesgue measure satisfies $0 < m(M) < m(\partial\mathcal{U})$. Our result gives a complete description of the class of holomorphic functions in $\mathcal{U}$ which are represented by the Carleman formulas on the open arc M, when $\partial\mathcal{U}$ is almost regular with respect to M (§2). That is, a type of integral representation formulas for functions holomorphic in a domain $\mathcal{U}$ by its values on a part M of the boundary $\partial\mathcal{U}$. This class is denoted by $\mathcal{NH}^1_M(\mathcal{U})$.

1. Introduction

Let $\mathcal{U}$ be a bounded, simply connected domain whose boundary $\partial\mathcal{U}$ is Jordan rectifiable curve and let M be an analytic open arc contained in $\partial\mathcal{U}$ whose Lebesgue measure satisfies $0 < m(M) < m(\partial\mathcal{U})$. Set $\overline{M} = M \cup \{a, b\}$ and define the function

$$u(x,y) = \begin{cases} 1 & \text{for } (x,y) \in M \\ 0 & \text{for } (x,y) \in \partial\mathcal{U} \setminus M. \end{cases} \tag{1.1}$$

Solving the corresponding Dirichlet problem we extend this function inside $\mathcal{U}$ as the harmonic function $\tilde{u}$. Then one obtains the holomorphic function

$$\phi(z) = \tilde{u}(z) + i\tilde{v}(z),\ z \in \mathcal{U}, \tag{1.2}$$

where $\tilde{v}(z)$ is the conjugate harmonic function of $\tilde{u}(z)$. The harmonic function $\tilde{u}(z)$ has angular boundary values almost everywhere, equal

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to 1 on M and to 0 on $\partial\mathcal{U} \setminus M$, that is $\tilde{u} = u$ a.e on the boundary of $\mathcal{U}$. Define the holomorphic function

$$\omega(z) = e^{\phi(z)},\ z \in \mathcal{U}. \tag{1.3}$$

The function ω has the following three properties 1) $|\omega(z)| = e$ a.e. for $z \in M$, 2) $|\omega(z)| = 1$, $z \in \partial\mathcal{U} \setminus M$ a.e., 3) $|\omega(z)| > 1$ for $z \in \mathcal{U}$. But $|\tilde{u}(z)| < 1$ in $\mathcal{U}$, hence $|\omega(z)| < e$ in $\mathcal{U}$. That is, ω belongs to the Smirnov class $H^\infty(\mathcal{U})$. In general, any element of $H^\infty(\mathcal{U})$ satisfying the properties 2) and 3) above is called **quenching function** [1]. Since M is free analytic (as an arc of a Jordan domain with rectifiable boundary) and $\omega(M) \subset \{w \in \mathbf{C} : \ |w| = 1\}$, we have that ω extends analytically across M (see [9] Theorem 13.4.8). This means that there exists domain V so that $M \subset V, \mathcal{U}^c \cap V \neq \emptyset$ and a holomorphic function $\hat{\omega} \in \mathcal{H}(V)$ satisfying $\omega(z) = \hat{\omega}(z)$, whenever $z \in \mathcal{U} \cap V$. For the rest of the paper we will not distinguish ω and $\hat{\omega}$. The following estimate which results from the reflection of $\omega(z)$ with respect to $\omega(M)$ will be of importance

$$|\omega(z)| > e,\ z \in \overline{\mathcal{U}}^c \cap V. \tag{1.4}$$

Recall (see [11]) that

Definition 1.1. *A function $f(z)$ holomorphic in $\mathcal{U}$ belongs to the class $E^p(\mathcal{U})$, $p > 0$, if there exists a sequence of curves Γ_m in $\mathcal{U}$ converging to $\partial\mathcal{U}$ (in a sense that it eventually surrounds every compact subdomain of $\mathcal{U}$) such that*

$$\int_{\Gamma_m} |f(z)|^p |dz| \leq C_1,$$

where C_1 is independent of m.

If $f \in E^1(\mathcal{U})$ then for every $z \in \mathcal{U}$, $n \in \mathbf{N}$, we have, by a theorem of Smirnov, that

$$f(z)\omega^n(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{U}} \frac{f(\zeta)\omega^n(\zeta)}{\zeta - z} d\zeta, \tag{1.5}$$

or equivalently

$$f(z) = \frac{1}{2\pi i} \int_{M} \frac{\omega^n(\zeta)}{\omega^n(z)} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\partial\mathcal{U}\setminus M} \frac{\omega^n(\zeta)}{\omega^n(z)} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{1.6}$$

where $\omega(z)$ is defined by (1.3).
Taking the limit $n \to \infty$ one has a variation of **Carleman formula**,

due to Goluzin-Krylov [13],

$$f(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_M \frac{\omega^n(\zeta)}{\omega^n(z)} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.7}$$

A posteriori, the convergence in (1.7) is uniform over the compact subsets of $\mathcal{U}$. Remark also that here we did not need the fact that ω has analytic continuation across M.

It is worth to note that one can obtain quenching functions by other means (ad-hoc) and not necessarily via the Dirichlet problem. Ad-hoc quenching functions were used in ([2], [3],[6]). This is not surprising, since neither Carleman integral representation formula, nor Cauchy integral representation formula are preserved under conformal mappings. The present setting of the problem is related to the one studied in [4], but the proof of the main theorem (see Theorem 2.1 and the proof of Lemma 3.5) is based on a 'local' analysis and on a geometric selection of adjacent points (see Definition 3.1). Actually it is the local consideration of analytic continuation of an analytic function that gives a full meaning to this class of points. This new approach applies to an essentially larger and much more general class of domains than the concrete examples already studied.

2. The class of functions representable by Carleman integral representation formula

If $\omega(z)$ is the quenching function defined by (1.3) then for every $0 < \tau < 1$ one defines the sets

$$\mathcal{U}_\tau = \{z \in \mathcal{U} : |\omega(z)| > 1 + \tau\}. \tag{2.1}$$

The open sets $\mathcal{U}_\tau$ are nonvoid, since for $z \in M$ we have that $|\omega(z)| = e$. Furthermore,

$$\begin{aligned} \partial\mathcal{U}_\tau &= \overline{M} \cup A_\tau, \text{ where,} \\ A_\tau &= \{z \in \mathcal{U} : |\omega(z)| = 1 + \tau\}. \end{aligned}$$

Let $\{\tau_n\}$ be a strictly decreasing sequence of positive numbers so that $\tau_n \longrightarrow 0$. We observe that $\partial\mathcal{U}_{\tau_n} \longrightarrow \partial\mathcal{U}$ and $\mathcal{U} = \cup_n \mathcal{U}_{\tau_n}$. That is, the sequence of open sets $\{\mathcal{U}\}_{\tau_n}$ forms an exhaustion of the domain $\mathcal{U}$. It is obvious that any other strictly decreasing to zero sequence $\{\tau'_n\}$ of positive numbers will give another exhaustion of the domain $\mathcal{U}$ which will be equivalent to the previous one, in the sense that for every open set belonging to one exhaustion there will be an open set belonging to the other one and containing it. However, the set $\overline{M}$ is the stable part of the boundary for every open set belonging to the exhaustion.

We claim that once $\partial\mathcal{U}$ is equipped with an essentially weak regularity condition (see ($\star$) below) then every A_τ, $0 < \tau < 1$ is an **Ahlfors regular curve**. This means that $l(A_\tau \cap D(\alpha, r)) \leq Cr$, where $D(\alpha, r)$ is a disk of radius r, $\alpha \in A_\tau$, l is the length of the curve $A_\tau \cap D(\alpha, r)$, and the constant C is independent of α.
If D is the unit disk, define the conformal transplant $\psi : D \longrightarrow \mathbf{R}_+$ by

$$\psi(w) := \tilde{u}(g^{[-1]}(w)),\ w \in D,$$

where $g(z)$ is the conformal map from $\mathcal{U}$ onto D (see§1). It follows then

$$\Delta\tilde{u}(z) = \Delta\psi(w)|g'(z)|,$$

where $\tilde{u}$ is the harmonic solution to (1.1) Hence ψ is the unique solution to the Dirichlet problem on the unit disk under the conditions $\psi(w) = 1,\ w \in g(M),\ \psi(w) = 0,\ w \in \partial D \setminus g(M)$. It was proved in [4] that the level curve $\{w \in D :\ \psi(w) = c\}$, $0 < c < 1$ is an arc of a circle joining the endpoints $g(a), g(b)$ of $g(M)$. Therefore the level curve $\{z \in \mathcal{U} :\ \tilde{u}(z) = c\}$ is $L_{ab} = g^{[-1]}(\{w \in D :\ \psi(w) = c\}$. Furthermore,

$$\overline{\{z \in \mathcal{U} : \tilde{u}(z) = c\}} \cap \partial\mathcal{U} = \{a, b\}.$$

It is not known in general when this level curve is Ahlfors regular so we proceed as follows: Riemann Mapping Theorem implies that there exists an onto conformal map $g : \mathcal{U} \longrightarrow D$, where D is the unit disk. Furthermore g extends as a homeomorphism between $\partial\mathcal{U}$ and ∂D. If $M \subset \partial\mathcal{U}$ is an open arc with endpoints $\partial M = \{a, b\}$ then we say that $\partial\mathcal{U}$ is **almost regular** with respect to M if the image under $g^{[-1]} : D \longrightarrow \mathcal{U}$ of the arc $l_{g(a),g(b)}$ in D, of any circle C intersecting ∂D at the points $g(a), g(b)$ is an Ahlfors regular (and particularly rectifiable) curve $L_{a,b}$ in $\mathcal{U}$ joining the points $a, b \in \partial\mathcal{U}$. One sufficient condition for the above type of boundary regularity is

$$\lim_{z \longrightarrow a} (g^{[-1]})'(z) = \alpha_1 \in \mathbf{C}, \text{ and } \lim_{z \longrightarrow b} (g^{[-1]})'(z) = \alpha_2 \in \mathbf{C}, \qquad (\star)$$

whenever z belongs to the cones (Stolz angles) containing the end points of the level curve. This forces the part of the level curve $L_{a,b}$ which is sufficiently 'close' to its its boundary points, to stay in a non-tangential region, and in particular to be rectifiable.

Regular polygons, with $\overline{M}$ contained in any side without its end-points, domains U for which the maps $(g^{[-1]})'$ are continuous in $\overline{D}$ (Kellogg's Theorem, p.426, [12]) or bounded, simply connected domains with piece-wise analytic boundary (with $\overline{M}$ contained in the interior of an analytic piece) are the basic examples of bounded, simply connected domains with Jordan rectifiable boundary which is almost

regular with respect M.
Thus, when M is an analytic arc, the sequence $\{\mathcal{U}_{\tau_n}\}$ is an Ahlfors-regular exhaustion of $\mathcal{U}$, in the sense that $\partial\mathcal{U}_{\tau_n}$ is an Ahlfors regular curve for every $n \in \mathbf{N}$. From the definition of the set $A_{\tau_n} = \partial\mathcal{U}_{\tau_n} \setminus \overline{M}$ one has that the following subordination property is valid:

$$\lim_{m\to\infty} |\frac{\omega^m(\zeta)}{\omega^m(z)}| \longrightarrow 0, \tag{2.2}$$

uniformly in $\zeta \in A_{\tau_n}$ whenever $z \in \mathcal{U}_{\tau_n}$ is fixed.
All the above lead to the following definition of a function space, independent of the Ahlfors-regular exhaustion of the domain $\mathcal{U}$.

Definition 2.1. *We say that a holomorphic function $f \in \mathcal{H}(\mathcal{U})$ with angular boundary values defined almost everywhere on M (denoted also by f) belongs to the Hardy class $\mathcal{H}^1$ near the base M and denote this class by $\mathcal{NH}^1_M(\mathcal{U})$, if $f \in E^1(\mathcal{U}_{\tau_n})$ for every $n \in \mathbf{N}$, where $\{\mathcal{U}_{\tau_n}\}_n$ is an Ahlfors-regular exhaustion of $\mathcal{U}$ defined for some strictly decreasing sequence of positive numbers $\{\tau_n\}_n$ which converges to 0.*

We now state the main theorem of the paper (in the spirit of results in [2], [3], [4], [6]).

Theorem 2.1. *Let $\mathcal{U}$ be bounded, simply connected domain with Jordan rectifiable boundary . Let also $M \subset \partial\mathcal{U}$ be an analytic arc satisfying $0 < m(M) < m(\partial\mathcal{U})$ and ω, defined by (1.3), be its quenching function. Let f be a holomorphic function in $\mathcal{U}$ having angular boundary values almost everywhere on M denoted also by f and satisfying $f \in L^1(M)$. If $\partial\mathcal{U}$ is almost regular with respect to M then for the Ahlfors regular exhaustion $\{\mathcal{U}_{\tau_n}\}$, defined by (2.1) one has*
1) If $f \in \mathcal{NH}^1_M(\mathcal{U})$ then the relation (1.7) holds and the convergence is uniform over the compact subsets of $\mathcal{U}$.
2) If (1.7) holds point-wise and in addition f has boundary values at the endpoints $\partial M = \{a, b\}$ then $f \in E^1(\mathcal{U}_\tau)$, $\tau > 0$, $\mathcal{U}_\tau \neq \emptyset$.

Proof of the first part: Since $f \in \mathcal{NH}^1_M(\mathcal{U})$, there exist Ahlfors-regular exhaustion $\{\mathcal{U}_{\tau_n}\}_n$ (constructed above) of $\mathcal{U}$. Each domain $\mathcal{U}_{\tau_n}$ is subordinated to the quenching function $\omega(z)$ defined above. Let $z \in \mathcal{U}$ be a fixed point. Then $z \in \mathcal{U}_{\tau_n}$ for some n. Hence $f \in E^1(\mathcal{U}_{\tau_n})$

and therefore by Cauchy formula we have, since $\overline{M} = M \cup \{a, b\}$,

$$f(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathcal{U}_{\tau_n}} \frac{\omega^m(\zeta)}{\omega^m(z)}\frac{f(z)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int\limits_{M} \frac{\omega^m(\zeta)}{\omega^m(z)}\frac{f(\zeta)d\zeta}{\zeta - z} \tag{2.3}$$

$$+ \frac{1}{2\pi i}\int\limits_{A_{\tau_n}} \frac{\omega^m(\zeta)}{\omega^m(z)}\frac{f(\zeta)d\zeta}{\zeta - z}. \tag{2.4}$$

From (2.2) and from the definition of A_{τ_n}, for every $n \in \mathbf{N}$ deduce that

$$\lim_{m\longrightarrow\infty} \frac{1}{2\pi i}\int\limits_{A_{\tau_n}} \frac{\omega^m(\zeta)}{\omega^m(z)}\frac{f(\zeta)d\zeta}{\zeta - z} = 0.$$

This implies that for every

$$f(z) = \lim_{m\to\infty} \frac{1}{2\pi i}\int\limits_{M} \frac{\omega^m(\zeta)}{\omega^m(z)}\frac{f(\zeta)d\zeta}{\zeta - z}.$$

The uniform convergence over the compact subsets follows. $\diamondsuit$

The rest of the proof will occupy the next section.

3. Proof of the second part of the Theorem 2.1

The simple observation that

$$\frac{\left(\frac{\omega(\zeta)}{\omega(z)}\right)^m}{\zeta - z} = \frac{1}{\zeta - z} + [\left(\frac{\omega(\zeta)}{\omega(z)}\right)^{m-1} + \left(\frac{\omega(\zeta)}{\omega(z)}\right)^{m-2} + \cdots + 1]\frac{\omega(\zeta) - \omega(z)}{\omega(z)(\zeta - z)}$$

for $z \in \mathcal{U}$ and $\zeta \in M$ leads to the following lemma (see also Lemma 3.1 in [4]), whose proof is direct.

Lemma 3.1. *Let f be a function holomorphic in $\mathcal{U}$ with the property that its boundary values on M belong to the class $L^1(M)$. If f is representable by* (1.7) *then pointwise*

$$f(z) = \frac{1}{2\pi i}\int\limits_{M} \frac{f(\zeta)d\zeta}{\zeta - z} + \sum_{m=0}^{\infty} \frac{g_m(z)}{\omega^{m+1}(z)},\ z \in \mathcal{U}, \tag{3.1}$$

where the functions

$$g_m(z) = \frac{1}{2\pi i}\int\limits_{M} f(\zeta)\omega^m(\zeta)\frac{\omega(\zeta) - \omega(z)}{\zeta - z}d\zeta,\ \textit{and}\ \omega^{m+1}(z)$$

are analytic in $\mathcal{U}$, for every $m = 0, 1, 2 \ldots$.

For $z \in \mathcal{U}$, set

$$F_+(z) = \frac{1}{2\pi i} \int\limits_M \frac{f(\zeta)}{\zeta - z} d\zeta$$

to be the Cauchy type integral. One observes that for every $z \in \mathcal{U}$ the function

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\omega^{m+1}(z)} = f(z) - F_+(z) \tag{3.2}$$

is holomorphic . This implies that for every $w_0 \in \mathcal{U}$ there exists an $r > 0$ so that

$$\mathcal{G}(z) = \sum_{j=0}^{\infty} \frac{\mathcal{G}(w_0)^{(j)}}{j!}(z - w_0)^j, \ z \in D(w_0, r).$$

Remark 1: We will show that for particular choices of w_0 this radius r (depending on w_0) is sufficiently large to extend $\mathcal{G}$ across the arc $\overline{\partial\mathcal{U}} \cap D(w_0, r) \neq \emptyset$. This and the monodromy theorem for holomorphic functions shall imply that G has a well defined analytic continuation across M.

Definition 3.1. *A point $\zeta_0 \in \mathcal{U}$ is called adjacent to M if 1) there exists $\rho_{\zeta_0} > 0$ so that the set $M_{\zeta_0,\rho_{\zeta_0}} = D(\zeta_0, \rho_{\zeta_0}) \cap M$ is non void, open arc, 2) $\overline{M}_{\zeta_0,\rho_{\zeta_0}} \subset M$.*

Observe that the first condition of the definition forces $\partial D(\zeta_0, \rho_{\zeta_0}) \cap M$ to be a two point set. This fact is of crucial importance in the constructions of Lemma 3.5. Note that since M is an analytic arc, adjacent points do exist. The following local division lemma is of importance.

Lemma 3.2. *Let $\zeta_0 \in \mathcal{U}$ be adjacent to the arc M. Then for $z \in D(\zeta_0, \rho_{\zeta_0})$ and $\zeta \in M \cap D(\zeta_0, \rho_{\zeta_0})$ one has*

$$\frac{\omega(z) - \omega(\zeta)}{z - \zeta} = \sum_{i=1}^{\infty} \frac{\omega(\zeta) - \sum\limits_{j=0}^{i-1} c_j (\zeta - \zeta_0)^j}{(\zeta - \zeta_0)^i} (z - \zeta_0)^{i-1}. \tag{3.3}$$

Proof: We begin by expressing $\omega(z)$ as a power series in $D(\zeta_0, \rho_{\zeta_0})$.

$$\omega(z) = \sum_{j=0}^{\infty} c_{\zeta_0,j} (z - \zeta_0)^j, \ z \in D(\zeta_0, \rho_{\zeta_0}).$$

This is possible, since the quenching function ω has analytic continuation across M and we implicitly assume that the domain of holomorphy of ω contains the disk $D(\zeta_0, \rho_{\zeta_0})$. Furthermore, after appropriate terms regrouping we get the desired result. The regrouping is possible because on one hand for $\zeta \in M \cap D(\zeta_0, \rho_{\zeta_0})$ the inequality $r_{\zeta_0} = \max\{|\zeta - \zeta_0|, |z - \zeta_0|\} < \rho_{\zeta_0}$ holds, and on the other hand

$$|\sum_{j=0}^{i-1}(z-\zeta_0)^{i-1-j}(\zeta-\zeta_0)^j| \leq \sum_{j=0}^{i-1}|(z-\zeta_0)^{i-1-j}(\zeta-\zeta_0)^j| \leq i(r_{\zeta_0})^{i-1}.$$

Combining this, together with the estimate

$$|c_{\zeta_0,j}| \leq \frac{A_{\zeta_0,\rho_{\zeta_0}}}{\rho^j_{\zeta_0}}, \tag{3.4}$$

where $A_{\zeta_0,\rho_{\zeta_0}} = \max_{\zeta\in\partial D(\zeta_0,\rho_{\zeta_0})}|\omega(\zeta)|$, one gets the desired result. $\diamondsuit$

Lemma 3.3. *Let $\zeta_0 \in \mathcal{U}$ be adjacent to M. Then, for every $m \in \mathbf{N}$ and $z \in D(\zeta_0, \rho_{\zeta_0})$ one has the equality*

$$g_m(z) = \frac{1}{2\pi i}\int_M f(\zeta)\omega^m(\zeta)\frac{\omega(\zeta)-\omega(z)}{\zeta - z}d\zeta = \sum_{j=0}^{\infty}\delta_{M,j,m}(z-\zeta_0)^j. \tag{3.5}$$

Proof: It is enough to show that the above expansion is valid for every $0 < \rho'_{\zeta_0} < \rho_{\zeta_0}$ so that the open arc $M'_{\zeta_0,\rho_{\zeta_0}} = D(\zeta_0, \rho'_{\zeta_0}) \cap M_{\zeta_0,\rho_{\zeta_0}} \neq \emptyset$. Actually, the integral

$$\frac{1}{2\pi i}\int_{M\setminus\overline{M'}_{\zeta_0,\rho_{\zeta_0}}} f(\zeta)\omega^m(\zeta)\frac{\omega(\zeta)-\omega(z)}{\zeta-z}d\zeta$$

is analytic function of z in the disk $D(\zeta_0, \rho'_{\zeta_0})$ and therefore can be written there as power series about ζ_0.
Fix $z \in D(\zeta_0, \rho'_{\zeta_0})$, then for every $\zeta \in \overline{M'}_{\zeta_0,\rho_{\zeta_0}}$ one has that

$$\frac{\omega(\zeta) - \sum\limits_{j=0}^{n-1} c_{\zeta_0,j}(\zeta-\zeta_0)^j}{(\zeta-\zeta_0)^n} = \sum_{l=0}^{\infty} c_{\zeta_0,n+l}(\zeta-\zeta_0)^l.$$

Using the estimate (3.4) we deduce that the inequality

$$|\sum_{l=0}^{\infty} c_{\zeta_0,n+l}(\zeta-\zeta_0)^l| \;\; \leq \;\; \frac{A_{\zeta_0,\rho_{\zeta_0}}}{\rho^n_{\zeta_0}}\frac{1}{(1-\frac{\rho'_{\zeta_0}}{\rho_{\zeta_0}})}$$

is valid for every $n = 1, 2, \ldots$. Therefore,

$$|\frac{1}{2\pi i}\int_{M'_{\zeta_0,\rho_{\zeta_0}}} f(\zeta)\omega^m(\zeta)\frac{\omega(\zeta)-\sum\limits_{j=0}^{n-1}c_{\zeta_0 j}(\zeta-\zeta_0)^j}{(\zeta-\zeta_0)^n}d\zeta| \leq C_{M'}\frac{A_{\zeta_0,\rho_{\zeta_0}}}{\rho^n_{\zeta_0}}\frac{1}{(1-\frac{\rho'_{\zeta_0}}{\rho_{\zeta_0}})}.$$

for some constant $C_{M'}$ which depents on $M'_{\zeta_0,\rho_{\zeta_0}}$. This implies the desired relation (3.5). Concluding, we observe that

$$\left(\limsup_n \sqrt[n]{|\delta_{M,n,m}|}\right)^{-1} \geq \rho_{\zeta_0} \ \diamondsuit$$

Let ζ_0 be an adjacent to M point. Since $\mathcal{G}$ is analytic in $\mathcal{U}\cap D(\zeta_0,\rho_0)$, can be expanded into a power series in disk about ζ_0, whose radius of convergence r_{ζ_0} is to be determined. More precisely,

$$\mathcal{G}(z)=\sum_{i=0}^{\infty}\frac{\mathcal{G}^{(i)}(\zeta_0)}{i!}(z-\zeta_0)^i,\ z\in D(\zeta_0,r_{\zeta_0}), \tag{3.6}$$

and

$$r_{\zeta_0}=\limsup_k \sqrt[k]{\frac{k!}{G^{(k)}(\zeta_0)}}.$$

Furthermore, in the disk $D(\zeta_0,\rho_{\zeta_0})$, for all $m=0,1,2\ldots$, one has that $\omega^{m+1}(z)=\sum\limits_{n=0}^{\infty}c_{\zeta_0,n,m}(z-\zeta_0)^n$, where the coefficients of the power series depend on m (see the proof of Lemma 3.2). Therefore, from Lemma 3.3,

$$\frac{g_m(z)}{\omega^{m+1}(z)} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}c_{\zeta_0,j,m+1}\delta_{M,n-j,m}\right)(z-\zeta_0)^n,\ z\in D(\zeta_0,\rho_{\zeta_0}). \tag{3.7}$$

From (3.2), (3.6) and (3.7) one has that

$$\mathcal{G}(\zeta_0)=\sum_{m=0}^{\infty}\frac{g_m(\zeta_0)}{\omega^{m+1}(\zeta_0)}=\sum_{m=0}^{\infty}\beta_{0,m},\ \text{where } \beta_{0,m}=c_{\zeta_0,0,m+1}\delta_{M,0,m}.$$

Again, by (3.7) and the point-wise equality for $z\in D(\zeta_0,\rho_{\zeta_0})$

$$\frac{\mathcal{G}(z)-\mathcal{G}(\zeta_0)}{z-\zeta_0}=\frac{1}{z-\zeta_0}\sum_{m=0}^{\infty}\left(\frac{g_m(z)}{\omega^{m+1}(z)}-\beta_{m,0}\right)$$

evaluated at $z=\zeta_0$, one deduces that

$$\mathcal{G}'(\zeta_0)=\sum_{m=0}^{\infty}\beta_{m,1},\ \text{where } \beta_{m,1}=\sum_{j=0}^{1}c_{m+1,\zeta_0,j}\lambda_{M,m,1-j}.$$

Repeating the same procedure for every $k=0,1,2,3\ldots$, it is elementary to show that

$$\frac{\mathcal{G}^{(k)}}{k!}(\zeta_0)=\sum_{m=0}^{\infty}\beta_{m,k},\ \text{where } \beta_{m,k}=\sum_{j=0}^{k}c_{m+1,\zeta_0,j}\lambda_{M,m,k-j}.$$

Equivalently, in the disk $D(\zeta_0, r_{\zeta_0}) \cap D(\zeta_0, \rho_{\zeta_0})$ one can interchange the order of summation just from point-wise convergence:

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\beta_{n,m}(z-\zeta_0)^n\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\beta_{n,m}\right)(z-\zeta_0)^n. \tag{3.8}$$

The crucial claim to verify is that $r_{\zeta_0} \geq \rho_{\zeta_0}$. This will be established in the Lemma 3.5 below. But before this we will need some more information about the behavior of the $\sum\limits_{m=0}^{\infty}\frac{g_m(z)}{\omega^{m+1}(z)}$ when $z \notin \mathcal{U}$.

Lemma 3.4. *Let ζ_0 be a point adjacent to M and let $0 < r < \rho_{\zeta_0}$ be such that $\partial D(\zeta_0, r) \cap \partial\mathcal{U} = \{b_0, b_0'\}$ with $b_0 \neq b_0'$. Assume that the function*

$$\mathcal{G}(z) = f(z) - \frac{1}{2\pi i}\int_M \frac{f(\zeta)}{\zeta - z}dz,\ z \in \mathcal{U},$$

has angular boundary values at the two distinct points b_0, b_0'. Then for every $z \in \partial D(\zeta_0, r) \cap \overline{\mathcal{U}}^c$ one has that

$$\sum_{m=0}^{\infty}\frac{g_m(z)}{\omega^{m+1}(z)} = -F_-(z), \tag{3.9}$$

where F_- denotes the exterior Cauchy integral

$$F_-(w) = \frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)d\zeta}{\zeta - w},\ w \in \overline{\mathcal{U}}^c.$$

Furthermore if α, β are two distinct points on

$$\partial D(\zeta_0, r) \cap (\overline{\mathcal{U}})^c$$

then one has the following estimate over the arc $l_{\alpha\beta}$

$$|\sum_{m=0}^{\infty}\int\limits_{l_{\alpha\beta}}\frac{\frac{g_m(z)}{\omega^{m+1}(z)}}{(z-\zeta_0)^{n+1}}dz| \leq \frac{c}{r^{n+1}}, \tag{3.10}$$

for some $c > 0$.

Proof: From Lemma 3.3, the analytic functions $\frac{g_m(z)}{\omega^{m+1}(z)}$ are well defined for every $m = 0, 1, 2, \ldots$ in the disk $D(\zeta_0, r)$ for all $0 < r < \rho_{\zeta_0}$. Furthermore the series of analytic functions

$$\lim_{m\longrightarrow\infty}\sum_{i=0}^{m-1}\int\limits_M f(\zeta)\frac{\omega^i(\zeta)}{\omega^{i+1}(z)}\frac{\omega(\zeta)-\omega(z)}{\zeta - z}d\zeta = \sum_{m=0}^{\infty}\int\limits_M f(\zeta)\frac{\omega^m(\zeta)}{\omega^{m+1}(z)}\frac{\omega(\zeta)-\omega(z)}{\zeta - z}d\zeta$$

converges absolutely in $z \in \partial D(\zeta_0, r) \cap \overline{\mathcal{U}}^c$. Actually, using properties of the quenching function and (1.4), one obtains, for every such z fixed, that $\frac{|\omega(\zeta)|}{|\omega(z)|} = \frac{e}{|\omega(z)|} < 1$, for all $\zeta \in M$. Thus for $z \in \partial D(\zeta_0, r) \cap \overline{\mathcal{U}}^c$ one has

$$\sum_{m=0}^{\infty} \int\limits_M f(\zeta) \frac{\omega^m(\zeta)}{\omega^{m+1}(z)} \frac{\omega(\zeta) - \omega(z)}{\zeta - w} d\zeta = \int\limits_M f(\zeta) (\sum_{m=0}^{\infty} \frac{\omega^m(\zeta)}{\omega^{m+1}(z)}) \frac{\omega(\zeta) - \omega(z)}{\zeta - z} d\zeta$$

$$= - \int\limits_M \frac{f(\zeta)}{\zeta - z} d\zeta = -F_-(z).$$

The second claim of the Lemma follows again from (1.4). For $z \in l_{\alpha\beta}$ there is a constant $\lambda > 1$ such that

$$|\frac{g_m(z)}{\omega^m(z)}| \leq A'_{max} e^{m(1-\lambda)} \int\limits_M |f(\zeta)| d\zeta,$$

where $|\frac{\omega(z)-\omega(\zeta)}{z-\zeta}| \leq A'_{max} = \frac{2C_{max}}{dist(l_{\alpha\beta}, M)}$ for the constant $C_{max} = \max_{z \in \bar{l}_{\alpha\beta}} |\omega(z)|$. This implies that

$$\sum_{m=0}^{\infty} \int\limits_{l_{\alpha\beta}} \frac{|\frac{g_m(z)}{\omega^{m+1}(z)}|}{|(z-\zeta_0)^{n+1}|} |dz| < \infty.$$

Thus for almost every $z \in l_{\alpha\beta}$ and for every $n \in \mathbf{N}$ one has

$$- \int\limits_{l_{\alpha\beta}} \frac{F_-(z)}{(z-\zeta_0)^{n+1}} dz = \sum_{m=0}^{\infty} \int\limits_{l_{\alpha\beta}} \frac{\frac{g_m(z)}{\omega^{m+1}(z)}}{(z-\zeta_0)^{n+1}} dz.$$

The relation (3.10) follows for $c = max_{z \in \bar{l}_{\alpha\beta}} |F_-(z)|$. $\diamondsuit$

Lemma 3.5. *Let $\zeta_0 \in \mathcal{U}$ be an adjacent point to M. Keeping the notation as above one has*

$$r_{\zeta_0} = \limsup_n \sqrt[n]{\frac{n!}{\mathcal{G}^{(n)}(\zeta_0)}} \geq \rho_{\zeta_0}. \tag{3.11}$$

Proof: Since ζ_0 is an adjacent point to M for almost all $0 < r < \rho_{\zeta_0}$, such that $D(\zeta_0, r) \cap \mathcal{U}^c \neq \emptyset$, the function

$$\mathcal{G}(z) = f(z) - \frac{1}{2\pi i} \int_M \frac{f(\zeta)}{\zeta - z} dz, \ z \in \mathcal{U},$$

has angular boundary values at the two points $\partial D(\zeta_0, r) \cap \partial \mathcal{U} = \{a_0, a'_0\}$. It is implicitly understood that these points depend on the adjacent point and the radius r. The existence of angular boundary values on M for the function $\mathcal{G}(z)$ follows from the identity $\mathcal{G}(z) = f(z) - F_+(z)$, $z \in \mathcal{U}$, and the fact that the Cauchy type integrals over Ahlfors-regular curves of L^1 functions belong to the Hardy class E^p, $0 < p < 1$ (see

[10]and [14]). Denote the angular boundary values of $\mathcal{G}$ at the points a_0, a_0' by $\mathcal{G}(a_0), \mathcal{G}(a_0')$ correspondingly. These coincide with minus the angular boundary values of the exterior Cauchy integral

$$F_-(w) = \frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)d\zeta}{\zeta - w},\ w \in \overline{\mathcal{U}}^c$$

almost everywhere. An extensive treatment and a detailed proof of this result follows from [7], [8], [10], [14]. (For the adaption to this setting see also [6].) Thus $-F_-$ is the pseudocontinuation of $\mathcal{G}$ on the arc $\partial D(\zeta_0, r) \cap \overline{\mathcal{U}}^c$. On this arc, and near the points a_0, a_0' we take two points d_0, d_0' correspondingly. Join the points a_0, a_0' by any smooth, non-tangent to $\partial\mathcal{U}$, simple curve l lying in $\mathcal{U} \cap D(\zeta_0, r)$ and leaving the point ζ_0 to the left. This curve divides the disk $D(\zeta_0, r)$ into two disjoint simply connected domains Ω_1, Ω_2 having Ahlfors-regular boundary. Assume that $\zeta_0 \in \Omega_1 \subset \mathcal{U}$. Then

$$\partial\Omega_1 = \big(\partial D(\zeta_0, r) \cap \overline{\mathcal{U}}\big) \cup l,\ \partial\Omega_2 = (\partial D(\zeta_0, r) \cap (\mathcal{U})^c) \cup l.$$

The fact that the function $\mathcal{G}$ has angular boundary values at the points a_0, a_0', implies that $\mathcal{G} \in H^\infty(\Omega_1)$. Therefore, for every $n \in \mathbf{N}$ one has that

$$\frac{\mathcal{G}^{(n)}(\zeta_0)}{n!} = \frac{1}{2\pi i}\int\limits_{\partial\Omega_1} \frac{\mathcal{G}(w)dw}{(w-\zeta_0)^{n+1}}.$$

This implies that

$$|\frac{1}{2\pi i}\int\limits_{\partial D(\zeta_0,r)\cap\overline{\mathcal{U}}} \frac{\mathcal{G}(w)dw}{(w-\zeta_0)^{n+1}}| \le \frac{M_r}{r^{n+1}}, \tag{3.12}$$

where $M_r = \max|\mathcal{G}(w)|$, $w \in \partial D(\zeta_0, r) \cap \overline{\mathcal{U}}$.

We modify the boundary of the domain Ω_1 by adding to it the arcs $l_{a_0d_0}, l_{a_0'd_0'}$ contained in $\partial D(\zeta_0, r) \cap \mathcal{U}^c$. As usual, the orientation of arcs is given by the order of its endpoints. Then for the union of arcs $\mathcal{L} = l_{d_0a_0} \cup (\partial D(\zeta_0, r) \cap \overline{\mathcal{U}}) \cup l_{a_0'd_0'} \cup l_{d_0'a_0'} \cup l \cup l_{a_0d_0}$ one has

$$\frac{\mathcal{G}^{(n)}(\zeta_0)}{n!} = \frac{1}{2\pi i}\int\limits_{\partial\Omega_1} \frac{\mathcal{G}(w)dw}{(w-\zeta_0)^{n+1}} = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{\mathcal{G}(w)dw}{(w-\zeta_0)^{n+1}},$$

where for the integration of $\mathcal{G}$ over the part of $\mathcal{L}$ outside the unit disk we use its pseudocontinuation. Thus,

$$|\frac{1}{2\pi i}\int\limits_{l_{d_0a_0}\cup(\partial D(\zeta_0,r)\cap\overline{\mathcal{U}})\cup l_{a_0'd_0'}} \frac{\mathcal{G}(w)dw}{(w-\zeta_0)^{n+1}}| \le \frac{M_r'}{r^{n+1}}, \tag{3.13}$$

where $M'_r = \max|\mathcal{G}(w)|$, $w \in l_{d_0a_0} \cup (\partial D(\zeta_0, r) \cap \overline{\mathcal{U}}) \cup l_{a'_0d'_0}$.
Next, we will show that over the part $\bar{l}_{d'_0a'_0} \cup l \cup \bar{l}_{a_0d_0}$ of $\partial\Omega_2$ one has similar estimate. Actually, the function $\frac{g_m(w)}{\omega^{m+1}(w)} \in \mathcal{H}(\overline{\Omega_2})$, for every $m \in \mathbf{N}$, since it is analytic in the disk $D(\zeta_0, \rho_{\zeta_0})$ containing properly the closure of Ω_2. Hence for every $m, n = 0, 1, 2, \dots$ the following holds:

$$\frac{1}{2\pi i}\int_{l_{d_0a_0}\cup\bar{l}\cup l_{a'_0d'_0}} \frac{\frac{g_m(w)}{\omega^{m+1}(w)}}{(w-\zeta_0)^{n+1}}dw = -\frac{1}{2\pi i}\int_{\bar{l}_{d'_0,d_0}} \frac{\frac{g_m(w)}{\omega^{m+1}(w)}}{(w-\zeta_0)^{n+1}}dw.$$

The fact

$$\sum_{m=0}^{\infty}\int_{\bar{l}_{d'_0d_0}} \frac{|\frac{g_m(w)}{\omega^{m+1}(w)}|}{|(w-\zeta_0)^{n+1}|}|dw| < \infty,$$

and Egoroff's theorem, imply that

$$\sum_{m=0}^{\infty}\int_{l_{d_0a_0}\cup\bar{l}\cup l_{a'_0d'_0}} \frac{\frac{g_m(w)}{\omega^{m+1}(w)}}{(w-\zeta_0)^{n+1}}dw = \int_{l_{d_0a_0}\cup\bar{l}\cup l_{a'_0d'_0}} \frac{\sum_{m=0}^{\infty}\frac{g_m(w)}{\omega^{m+1}(w)}}{(w-\zeta_0)^{n+1}}dw.$$

Thus, using lemma 3.4, the above mention fact about of the absolute convergence of the series, and since $-F_{-}(w) = \mathcal{G}(w)$ for every $w \in \partial D(\zeta_0, \rho_{\zeta_0}) \cap \mathcal{U}^c$ we obtain

$$|\int_{\bar{l}_{d_0a_0}\cup l\cup\bar{l}_{a'_0d'_0}} \frac{\mathcal{G}(w)}{(w-\zeta_0)^{n+1}}dw| \leq \frac{c}{r^{n+1}}, \tag{3.14}$$

for some constant $c > 0$. Combining $(3.13), (3.14)$ we obtain

$$|\frac{\mathcal{G}^{(n)}(\zeta_0)}{n!}| \leq \frac{c'}{r^{n+1}},$$

where $c' > 0$ is a suitable constant. It follows that the corresponding power series of $\mathcal{G}(z)$ about ζ_0 has radius of convergence at least r. But $r \longrightarrow \rho_{\zeta_0}$ from below, thus we get that the radius of convergence for the power series of $\mathcal{G}(z)$ about ζ_0 is at least equal to ρ_{ζ_0}. Thus (3.6) is valid in the disk $D(\zeta_0, \rho_{\zeta_0})$ and hence

$$\mathcal{G}(z) = \sum_{n=0}^{\infty}\frac{\mathcal{G}^{(n)}(\zeta_0)}{n!}(z-\zeta_0)^n, \ z \in D(\zeta_0, \rho_{\zeta_0}). \ \diamondsuit$$

Remark 2: The last lemma implies that minus the exterior Cauchy type integral which is equal to

$$-F_{-}(z) = -\frac{1}{2\pi i}\int\limits_{M} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{3.15}$$

whenever $z \notin \overline{\mathcal{U}}$ can be continued analytically through the open arc M. This follows from the monodromy theorem, the fact that $\mathcal{G}(z) = -F_{-}(z)$ almost everywhere on M and that $\mathcal{G}(z)$ has analytic continuation across the arc M. (See also Remark 1).
The final lemma (originally found in [4], but also with complete proof in [6]) that we will need is the following

Lemma 3.6. *Let Ω be a bounded, simply connected domain with Ahlfors-regular boundary. Let $\Gamma \subset \partial\Omega$ be a curve, whose length $l(\Gamma)$ is strictly smaller than $l(\partial\Omega)$. If Ω' is any simply connected sub-domain of Ω with Ahlfors regular boundary containing a curve $\gamma \subset \Gamma$ then the Cauchy type integral*

$$h(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\phi(t)}{t - z} dt,\ z \in \Omega,\ \phi \in L^1(\Gamma)$$

belongs to the class $E^p(\Omega')$ for every $0 < p < 1$.

At this point we conclude the proof of the second part the Theorem 2.1.
Proof: Lemma 3.1 implies that in order to prove the second part of Theorem 2.1 one needs first to show that the holomorphic function, defined as a point-wise limit,

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\omega^{m+1}(z)} = f(z) - \frac{1}{2\pi i}\int\limits_{M} \frac{f(\zeta)}{\zeta - z} d\zeta,\ z \in \mathcal{U}$$

can be continued analytically across the open arc M. This follows from the already established (see lemmas $3.2, 3.3, 3.4, 3.5$) fact that $\mathcal{G}$ has analytic continuation to $D(\zeta_0, \rho_{\zeta_0})$ for every ζ_0 adjacent to M and from Remark 1.

Since by assumption f has angular boundary values at $\partial M = \{a, b\}$, it is straightforward to show that

$$|\mathcal{G}(z)| \leq C + |\int\limits_{M} \frac{f(\zeta) d\zeta}{\zeta - z}|,\ z \in \mathcal{U}_{\tau}, \tag{3.16}$$

for all $0 < \tau < 1$. Furthermore, Lemma 3.6 implies that the function

$$F_+(z) = \frac{1}{2\pi i} \int\limits_M \frac{f(\zeta)d\zeta}{\zeta - z}, \ z \in \mathcal{U},$$

being a Cauchy type integral, belongs to the Smirnov class $E^p(\mathcal{U}_\tau)$ for every $0 < p < 1$. Therefore from 3.16 and the fact that $\mathcal{G}$ has analytic continuation across M, given by $-F_-$, one has that for every $0 < p < 1$

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\omega^{m+1}(z)} \in E^p(\mathcal{U}_\tau)$$

for every $\tau > 0$, $\tau \longrightarrow 0$ and for every domain $\mathcal{U}_\tau$ defined in §2. Therefore, see Remark 2, for every $0 < p < 1$, the function

$$f(z) = F_+(z) + \mathcal{G}(z) = F_+(z) - F_-(z) \in E^p(\mathcal{U}_\tau) \tag{3.17}$$

for every $\mathcal{U}_\tau$. Again, the assumption about f having boundary values at the endpoints a, b of M implies that $f \in L^\infty(A_\tau)$. That is, $f \in L^1(\partial\mathcal{U}_\tau)$. Thus $f \in E^1(\mathcal{U}_\tau)$ for every $\tau > 0$, $\tau \longrightarrow 0$. This follows by a theorem of V.Smirnov stating that if a function $f \in E^p$ and its boundary values $f \in L^q$, $p < q$, then $f \in E^q$ ([14]). $\diamondsuit$

Remark 3: We would like to mention that the analogous to the Theorem 2.1 result, holds in the case of Vector and Operator-valued holomorphic functions. The proof in this case is essentially the same, provided that we use the setting and analogous results as presented in [5].

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