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J. Math. Anal. Appl. 310 (2005) 657–672
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On a class of holomorphic functions representable by Carleman formulas in the interior of an equilateral cone from their values on its rigid base

George Chailos [a], Alekos Vidras [b,*]

[a] *Department of Computer Science, Intercollege, Nicosia 1700, Cyprus*
[b] *Department of Mathematics and Statistics, University of Cyprus, Nicosia 1678, Cyprus*

Received 7 April 2004
Available online 25 March 2005
Submitted by M. Passare

Abstract

Let Δ be an equilateral cone in **C** with vertices at the complex numbers $0, z_1^0, z_2^0$ and rigid base M (Section 1). Assume that the positive real semi-axis is the bisectrix of the angle at the origin. For the base M of the cone Δ we derive a Carleman formula representing all those holomorphic functions $f \in \mathcal{H}(\Delta)$ from their boundary values (if they exist) on M which belong to the class $\mathcal{NH}_M^1(\Delta)$. The class $\mathcal{NH}_M^1(\Delta)$ is the class of holomorphic functions in Δ which belong to the Hardy class $\mathcal{H}^1$ *near* the base M (Section 2). As an application of the above characterization, an important result is an extension theorem for a function $f \in L^1(M)$ to a function $f \in \mathcal{NH}_M^1(\Delta)$.

Keywords: Carleman formula; Cone with a rigid base

* Corresponding author.
E-mail addresses: chailos.g@intercollege.ac.cy (G. Chailos), msvidras@ucy.ac.cy (A. Vidras).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.02.036

1. Introduction

Three noncollinear points $0, z_1^0, z_2^0$ in the complex plane define a cone Δ with vertex at the origin whose width is $\frac{\pi}{n}$, $n = 2, 3, 4, \ldots$. The cone Δ is equilateral if $|z_1^0 - 0| = |z_2^0 - 0|$. Any simple, Ahlfors regular curve M, with parametrization $\zeta(t)$, connecting the vertices z_1^0, z_2^0, generating simply connected bounded region, is called base of the cone Δ. For small $\epsilon > 0$ the set $E_{M,\epsilon} = (M \cap D(z_1^0, \epsilon)) \cup (M \cap D(z_2^0, \epsilon))$ is called the ϵ-end set of M (we implicitly assume here that $D(z_1^0, \epsilon) \cap D(z_2^0, \epsilon) = \emptyset$). The ϵ-end set of M is called flat if the slope $\arg \zeta'(t)$ is a monotone (increasing or decreasing) function of the parameter t. The base M is called rigid if it has flat ϵ-ends for some $\epsilon > 0$ and satisfies the following two conditions:

(i) $M \subset D(x_M, R_M)$, where $x_M = M \cap (0, +\infty)$ and $R_M = |x_M - z_1^0|$,
(ii) $\Re z_1^0 < \max_{\zeta \in \bar{M}} \Re \zeta = x_M$.

Geometrically the above conditions mean that M cannot approach the origin too close, that M cannot be a segment, that circles, centered at x_M and of radius $|\zeta - x_M|$, $\zeta \in E_{M,\epsilon}$, intersect M at exactly two points. We assume, for reasons of simplicity, that the positive real semi-axis is the bisectrix of the angle $\widehat{z_1^0 0 z_2^2}$ at the origin. From now on we assume that Δ *denotes an equilateral cone with vertex at the origin and rigid base* M. Define the holomorphic function

$$\phi(z) = e^{z^n}, \quad z \in \Delta. \tag{1.1}$$

The function ϕ is holomorphic in a neighborhood of Δ and has the following two properties:

(1) $|\phi(z)| = 1$ a.e. for $z \in \partial\Delta \setminus M$,
(2) $|\phi(z)| > 1$ for $z \in \Delta$.

Actually, if z belongs to the side $|z_0^1|$, $\arg z = \frac{\pi}{2n}$ of the cone Δ, then $\exp(z^n) = \exp(r^n \cos(n\theta) + ir^n \sin(n\theta))$, $0 \leqslant r \leqslant |z_1^0|$, $\theta = \frac{\pi}{2n}$. This implies that $|\exp(z^n)| = \exp(r^n \cos(n\theta)) = \exp(r^n \cos(n\frac{\pi}{2n})) = 1$. Similarly, one can show that $|\exp(z^n)| = 1$, whenever

$$z = re^{-i\frac{\pi}{2n}}, \quad 0 \leqslant r \leqslant |z_2^0|.$$

If $z \in \Delta$ then $-\frac{\pi}{2n} < \theta < \frac{\pi}{2n}$. It follows from this that $|\exp(z^n)| > 1$. The properties (1) and (2) above and the fact that $\phi \in H^\infty(\Delta)$ characterize this function as a quenching function off the side M for the cone Δ [1–4].

Let $f \in E^1(\Delta)$ (see Definition 2.1); then for every $z \in \Delta$ and $m \in \mathbf{N}$ we have, by a theorem of V. Smirnov (1932), that

$$f(z) = \frac{\phi^{-m}(z)}{2\pi i} \int_{\partial\Delta} \frac{f(\zeta)\phi^m(\zeta)}{\zeta - z} \, d\zeta \tag{1.2}$$

$$= \frac{1}{2\pi i} \int\limits_M \frac{f(\zeta)(\frac{\phi(\zeta)}{\phi(z)})^m}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int\limits_{\partial\Delta\setminus M} \frac{f(\zeta)(\frac{\phi(\zeta)}{\phi(z)})^m}{\zeta - z} \, d\zeta, \tag{1.3}$$

where $\phi(z)$ is defined by (1.1).

Taking the limit $m \to \infty$ one has a variation of the original *Carleman formula* [1],

$$f(z) = \lim_{m\to\infty} \frac{1}{2\pi i} \int\limits_M \frac{f(\zeta)(\frac{\phi(\zeta)}{\phi(z)})^m}{\zeta - z} \, d\zeta. \tag{1.4}$$

A posteriori, the convergence in (1.4) is uniform over the compact subsets of Δ.

We remark here that the quenching function is not unique. On one hand it is always possible to obtain it by solving the corresponding Dirichlet problem, on the other hand this approach is not explicit enough (see however [4]). Sometimes, one can obtain the quenching function ad-hoc, as is the case here. Other such cases (and their multidimensional analogues) were studied in [2,3].

The first sections of the present paper are devoted to the description of the class of holomorphic functions in the cone Δ which are representable by the Carleman integral representation formula (1.4). The crucial Lemma 3.6 is a nontrivial refinement of the similar argument in [4].

As a main application we will state an extension theorem (see [9]). In general, one is looking to find necessary and sufficient conditions for a function $f \in L^1(M)$ to be extendable into an analytic function in Δ, belonging to the class $\mathcal{NH}^1_M(\Delta)$ (Section 2). Such results were obtained in [2–4] for other types of simply connected domains. However, every separate case seems so distinct that until now we were not able to formulate a general type of theorem to cover all the cases.

2. The class of functions representable by Carleman integral representation formula

For the above cone Δ and the quenching function ϕ defined by (1.1) we consider the domains

$$\Delta_\tau = \left\{ z \in \Delta \colon \left|\phi(z)\right| > 1 + \tau \right\}, \tag{2.1}$$

where $\tau > 0$, $\tau \to 0$. Geometrically, the domains Δ_τ are bounded by hyperbolas $r^n \cos(n\theta) = \ln(1+\tau)$, $\theta = \pm\frac{\pi}{2n}$, and the base M. The vertex of the hyperbola is at the point $\zeta_{0,\tau} = (\ln|1+\tau|)^{\frac{1}{n}}$. Furthermore, if

$$\begin{aligned} M_\tau &= M \cap \left\{ z \in \mathbf{C} \colon \left|\phi(z)\right| \geqslant 1 + \tau \right\}, \\ A_\tau &= \left\{ z \in \Delta \colon \left|\phi(z)\right| = 1 + \tau \right\} \end{aligned} \tag{2.2}$$

then $\partial\Delta_\tau = M_\tau \cup A_\tau$.

Let $\{\tau_n\}$ be a decreasing sequence of positive numbers converging to 0. If $\{\Delta_{\tau_n}\}$ is the sequence of the domains defined as in (2.1) then it is an Ahlfors regular exhaustion of the domain Δ attached to the base M. By Ahlfors regular exhaustion of the cone Δ attached to the base M we mean that the sequence of the domains $\{\Delta_{\tau_n}\}$ is increasing, that is $\Delta_{\tau_n} \subset \Delta_{\tau_{n+1}}$ for every $n \in \mathbf{N}$, $\partial\Delta_{\tau_n} \to \partial\Delta$, $\partial\Delta_{\tau_n} \cap \partial\Delta = M_{\tau_n} \subset M$ and satisfies

(1) $\Delta = \bigcup_n \Delta_{\tau_n}$,
(2) the boundary $\partial\Delta_{\tau_n}$ is an Ahlfors regular curve, that is, $l(\partial\Delta_{\tau_n} \cap D(b,r)) \leqslant Cr$, where $D(b,r)$ is a disk of radius r, and center $b \in \partial\Delta_{\tau_n}$, l is the length of the curve $\partial\Delta_{\tau_n} \cap D(b,r)$, and the constant C is independent of b [4].

Furthermore, there is a sense of subordination of the exhaustion $\{\Delta_{\tau_n}\}$ of the domain Δ to the quenching function ϕ off the set M. To be more precise, one has that for every $z \in \Delta_{\tau_n}$ fixed, $\lim_{m\to\infty} \frac{|\phi^m(\zeta)|}{|\phi^m(z)|} = 0$ uniformly in $\zeta \in \partial\Delta_{\tau_n} \setminus M_{\tau_n}$. Indeed, if $\zeta \in \partial\Delta_{\tau_n} \setminus M_{\tau_n}$ then $|\phi(\zeta)| = 1 + \tau_n$. On the other hand, from the definition of Δ_{τ_n} we have $|\phi(z)| > 1 + \tau_n$ whenever $z \in \Delta_{\tau_n}$.

Next, we recall the following

Definition 2.1. A function $f(z)$ holomorphic in Δ belongs to the class $E^p(\Delta)$, $p > 0$, if there exists a sequence of curves γ_m in Δ converging to $\partial\Delta$, in the sense that $\{\gamma_m\}$ eventually surrounds each compact sub-domain of Δ, such that

$$\int_{\gamma_m} \left|f(z)\right|^p |dz| \leqslant C_1,$$

where C_1 is independent of m.

Hence, if a function $h \in E^1(\Delta_{\tau_n})$ then h has angular boundary values, denoted also by h, almost everywhere on $\partial\Delta_{\tau_n}$ and

$$\sup_m \int_{\gamma_{m,n}} \left|h(z)\right| |dz| < \infty,$$

where $\gamma_{m,n}$ are rectifiable curves converging to $\partial\Delta_{\tau_n}$. Now we are ready to introduce the class of holomorphic functions that belong to the Hardy class $\mathcal{H}^1$ near the base M [2–4].

Definition 2.2. We say that a holomorphic function $f \in \mathcal{H}(\Delta)$ with angular boundary values defined almost everywhere on M (denoted also by f) belongs to the Hardy class $\mathcal{H}^1$ near the base M and denote this class by $\mathcal{NH}^1_M(\Delta)$ if $f \in E^1(\mathcal{W}_n)$, where $\{\mathcal{W}_n\}_n$ is an Ahlfors regular exhaustion of Δ attached to M.

Remark 2.1. It is clear that the above definition does not depend on Ahlfors regular exhaustion of Δ attached to M [4]. This allows us to consider the particular Ahlfors regular exhaustion of $\{\Delta_{\tau_n}\}$ of Δ attached to M and subordinated to the quenching function ϕ. This approach was already used in [2–4].

Example 3.1 will show that $\mathcal{NH}^1_M(\Delta) \neq E^1(\Delta)$.

Next we state a characterization theorem (in the spirit of results in [2–4]).

Theorem 2.1. *Let M be the base of the cone Δ opposite to the vertex at the origin and ϕ, defined by* (1.1)*, be its quenching function. Let f be a holomorphic function in the cone Δ having angular boundary values almost everywhere on M denoted also by f and satisfying $f \in L^1(M)$. Then for the Ahlfors regular exhaustion $\{\Delta_{\tau_n}\}$ defined by* (2.1) *one has*:

(1) *If* $f \in \mathcal{NH}^1_M(\Delta)$ *then the relation* (1.4) *holds and the convergence is uniform over the compact subsets.*
(2) *If* (1.4) *holds point-wise then* $f \in E^1(\Delta_\tau)$, $\tau > 0$, $\Delta_\tau \neq \emptyset$.

Proof of the first part. Since $f \in \mathcal{NH}^1_M(\Delta)$, we take the Ahlfors regular exhaustion $\{\Delta_{\tau_n}\}_n$ (constructed above) of Δ which is attached to the base M and is subordinated to the quenching function $\phi(z) = e^{z^n}$. Let $z \in \Delta$ be a fixed point. Then $z \in \Delta_{\tau_n}$ for some n. Hence $f \in E^1(\Delta_{\tau_n})$ and therefore by Cauchy formula we have

$$f(z) = \frac{1}{2\pi i}\int\limits_{M_{\tau_n}} f(\zeta)\frac{\phi^m(\zeta)}{\phi^m(z)}\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{A_{\tau_n}} f(\zeta)\frac{\phi^m(\zeta)}{\phi^m(z)}\frac{d\zeta}{\zeta - z}, \tag{2.3}$$

where $M_{\tau_n} = M \cap \partial\Delta_{\tau_n}$. The second integral tends to 0 as $m \to \infty$ because the exhaustion $\{\Delta_{\tau_n}\}_n$ is subordinated to the quenching function ϕ. Thus

$$\begin{aligned}
&\lim_{m\to\infty}\frac{1}{2\pi i}\int\limits_{M_{\tau_n}} f(\zeta)\phi^m(\zeta)\phi^{-m}(z)\frac{d\zeta}{\zeta - z}\\
&\quad = \lim_{m\to\infty}\frac{1}{2\pi i}\left(\int\limits_{M} f(\zeta)\frac{\phi^m(\zeta)}{\phi^m(z)}\frac{d\zeta}{\zeta - z} - \int\limits_{M\setminus M_{\tau_n}} f(\zeta)\frac{\phi^m(\zeta)}{\phi^m(z)}\frac{d\zeta}{\zeta - z}\right).
\end{aligned}$$

We claim that the second integral tends to 0 as $m \to \infty$. Indeed, it is enough to show that for every $z \in \Delta_{\tau_n}$ fixed, one has $\lim_{m\to\infty}|\frac{\phi^m(\zeta)}{\phi^m(z)}| = 0$ for every $\zeta \in M \setminus M_{\tau_n}$. To show this, one observes from the construction of the exhaustion, since $\zeta \in M \setminus M_{\tau_n}$, that $\zeta \in \partial\Delta_{\tau'_n}$ for some $\tau'_n < \tau_n$. Hence, $|\phi(z)| > 1 + \tau_n > 1 + \tau'_n = |\phi(\zeta)|$. Thus, the Carleman integral representation formula (1.4) holds. The uniform convergence over the compact subsets follows. This concludes the proof of the first part. □

The rest of the proof will occupy the next section.

3. Proof of the second part of Theorem 2.1

Our first step is to rewrite (1.4) in an equivalent way. We observe that

$$\frac{(\frac{\phi(\zeta)}{\phi(z)})^m}{\zeta - z} = \frac{1}{\zeta - z} + \left[\left(\frac{\phi(\zeta)}{\phi(z)}\right)^{m-1} + \left(\frac{\phi(\zeta)}{\phi(z)}\right)^{m-2} + \cdots + 1\right]\frac{\phi(\zeta) - \phi(z)}{\phi(z)(\zeta - z)}. \tag{3.1}$$

The relation (3.1) implies the following

Lemma 3.1. *Let* f *be a function holomorphic in the cone* Δ *with the property that its boundary values on* M *belong to the class* $L^1(M)$. *If* f *is representable by* (1.4) *then point-wise*

$$f(z) = \frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)\,d\zeta}{\zeta - z} + \sum_{m=0}^{\infty}\frac{g_m(z)}{\phi^{m+1}(z)}, \quad z \in \Delta, \tag{3.2}$$

where the functions

$$g_m(z) = \frac{1}{2\pi i} \int_M f(\zeta)\phi^m(\zeta) \frac{\phi(\zeta) - \phi(z)}{\zeta - z} \, d\zeta, \quad z \in \Delta,$$

are analytic in Δ *for every* $m = 0, 1, 2, \ldots$.

Keeping the notation of the above lemma, observe that for every $z \in \Delta$ the function

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\phi^{m+1}(z)} = f(z) - \frac{1}{2\pi} \int_M \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

is holomorphic in Δ. This implies that for every $w_0 \in \Delta$ there exists an $r > 0$ so that

$$\mathcal{G}(z) = \sum_{i=0}^{\infty} \frac{\mathcal{G}(w_0)^{(i)}}{i!} (z - w_0)^i, \quad z \in D(w_0, r).$$

We will show that for particular choices of w_0 (which are sufficiently close to x_M) this radius r is sufficiently large to extend $\mathcal{G}$ across the base M. That is, we will prove that base $M = D(w_0, r) \cap \partial\Delta$ and therefore $\mathcal{G}$ has analytic continuation across the arc M.

In order to accomplish this we will follow the approach developed in [3]. As a first step, we expand into power series in $z - w_0$ the functions $\frac{g_m(z)}{\phi^{m+1}(z)}$, $m = 0, 1, 2, \ldots$, within the same disk, whose center will be chosen conveniently. Then in (3.2) one has series of power series and the main difficulty is to interchange the order of summation (no known type conditions are in generally present).

Definition 3.1. A point $w_0 \in (0, \infty) \cap \Delta$ is called *adjacent* to M if for every $\zeta \in E_{M,\epsilon}$ the circle centered at w_0 and radius $|w_0 - \zeta|$ intersects M at exactly two points.

Remark 3.1. The above condition forces the adjacent points to be sufficiently close to x_M. This condition will be used in the proof of Lemma 3.6.

Lemma 3.2. *For the cone* Δ, *let* w_0 *be adjacent to* M. *Then, for every* $m \in \mathbf{N}$, *the series*

$$\phi^m(z) = e^{mz^n} = \sum_{i=0}^{\infty} \gamma_{m,w_0,i} (z - w_0)^i,$$

converges, whenever $z \in D(w_0, R_{w_0})$, *where the radius* R_{w_0} *is independent of* m *and is equal to* $R_{w_0} = |w_0 - z_1^0|$. *Furthermore,*

$$|\gamma_{m,w_0,i}| \leqslant \frac{M_{m,R_{w_0}}}{R_{w_0}^i}, \tag{3.3}$$

where $M_{m,R_{w_0}} = \max_{z \in \bar{D}(w_0, R_{w_0})} |\phi^m(z)|$.

Proof. The function $\phi^m(z)$ is entire for every $m = 0, 1, 2, \ldots$. $\square$

Remark here that for $m = 1$ the Taylor coefficients $\gamma_{1,w_0,i}$ of the above power series correspond the Taylor coefficients of the power series expansion about the point w_0 of the function $\phi(z) = e^{z^n}$. The next lemma expresses the local division property.

Lemma 3.3. *For the cone Δ, let $w_0 \in \Delta$ be adjacent to the base M. Then for $z \in D(w_0, R_{w_0})$ and $\zeta \in M$ one has*

$$\frac{\phi(z) - \phi(\zeta)}{z - \zeta} = \sum_{i=1}^{\infty} \frac{\phi(\zeta) - \sum_{j=0}^{i-1} \gamma_{1,w_0,j}(\zeta - w_0)^j}{(\zeta - w_0)^i}(z - w_0)^{i-1}. \tag{3.4}$$

Proof. Recall that from the previous lemma we have that for every $z \in D(w_0, R_{w_0})$ the following expansion is valid: $\phi(z) = \sum_{j=0}^{\infty} \gamma_{1,w_0,j}(z - w_0)^j$. Furthermore, one has for every $\zeta \in M \subset D(w_0, R_{w_0})$ that

$$\begin{aligned}
\frac{\phi(z) - \phi(\zeta)}{z - \zeta} &= \sum_{i=0}^{\infty} \gamma_{1,w_0,i} \frac{((z - w_0)^i - (\zeta - w_0)^i)}{z - \zeta} \\
&= \frac{1}{z - \zeta}\left(\sum_{i=0}^{\infty}(z - \zeta)\gamma_{1,w_0,i}\left(\sum_{j=0}^{i-1}(z - w_0)^{i-1-j}(\zeta - w_0)^j\right)\right) \\
&= \frac{\phi(\zeta) - \gamma_{1,w_0,0}}{\zeta - w_0} + \frac{\phi(\zeta) - \sum_{j=0}^{1} \gamma_{1,w_0,j}(\zeta - w_0)^j}{(\zeta - w_0)^2}(z - w_0) + \cdots \\
&\quad + \frac{\phi(\zeta) - \sum_{j=0}^{i-1} \gamma_{1,w_0,j}(\zeta - w_0)^j}{(\zeta - w_0)^i}(z - w_0)^{i-1} + \cdots,
\end{aligned} \tag{3.5}$$

after regrouping. The regrouping is possible because on one hand for $\zeta \in M$ the inequality $r_{w_0} = \max\{|\zeta - w_0|, |z - w_0|\} < R_{w_0}$ implies

$$\left|\sum_{j=0}^{i-1}(z - w_0)^{i-1-j}(\zeta - w_0)^j\right| \leqslant \sum_{j=0}^{i-1}\left|(z - w_0)^{i-1-j}(\zeta - w_0)^j\right| \leqslant i(r_{w_0})^{i-1}$$

and on the other hand one has the relation (3.3). □

We state the following lemma, without proof since its proof is elementary.

Lemma 3.4. *For the cone Δ, let $w_0 \in \Delta$ be adjacent to the base M. Then, for every $m \in \mathbf{N}$ and $z \in D(w_0, R_{w_0})$ one has the equality*

$$g_m(z) = \frac{1}{2\pi i} \int_M f(\zeta)\phi^m(\zeta) \frac{\phi(\zeta) - \phi(z)}{\zeta - z}\, d\zeta = \sum_{i=0}^{\infty} \lambda_{M,m,i}(z - w_0)^i. \tag{3.6}$$

The following lemma we need is taken from [4]. We include the proof for reason of completeness.

Lemma 3.5. *Let Ω be a bounded, simply connected domain with Ahlfors regular boundary. Let $\Gamma \subset \partial\Omega$ be a curve, whose length $l(\Gamma)$ is strictly smaller than $l(\partial\Omega)$. If Ω' is any simply connected sub-domain of Ω with Ahlfors regular boundary containing a curve $\gamma \subset \Gamma$ then the Cauchy type integral*

$$h(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(t)}{t - z}\, dt, \quad z \in \Omega,\ \phi \in L^1(\Gamma),$$

belongs to the class $E^p(\Omega')$ for every $0 < p < 1$.

Proof. Since the boundaries of the domains Ω, Ω' and $\Omega \setminus \bar{\Omega}'$ are Ahlfors regular, Smirnov theorem holds, that is, the Cauchy type integrals over Γ, γ and over $\Gamma \setminus \gamma$ of the L^1-function ϕ belong to the corresponding Hardy class E^p for every $0 < p < 1$. The original result was proven for the unit disk by V. Smirnov (1928) [10, Theorem 3.5], but was extended to any simply connected domain with Ahlfors regular boundary in [5–7].

If for $\phi \in L^1(\Gamma)$ and $z \in \Omega$,

$$h(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(t)}{t - z}\, dt = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(t)}{t - z}\, dt + \frac{1}{2\pi i} \int_{\Gamma\setminus\gamma} \frac{\phi(t)}{t - z}\, dt = h_1(z) + h_2(z),$$

then $h_1 \in E^p(\Omega')$ for every $0 < p < 1$. Furthermore, it is clear that for every $0 < p < 1$ one has that $h_2 \in E^p(\Omega) \cap E^p(\Omega \setminus \bar{\Omega}')$. Let $\{\Omega_n\}_n$ be a sequence of domains such that $\Omega_n \subset \Omega_{n+1}$, $\partial\Omega_n \mapsto \partial\Omega$ and $\int_{\partial\Omega_n} |h_2(z)|^p\, |dz| \leqslant M_1 < \infty$. Such a sequence of domains exist from the fact that $h_2 \in E^p(\Omega)$. Thus, it follows with the aid of Theorem 10.3 in [8] that

$$\int_{\partial(\Omega_n \setminus \bar{\Omega}'_n)} \left|h_2(z)\right|^p |dz| \leqslant M_2 < \infty,$$

where $\Omega'_n = \Omega' \cap \Omega_n$ for every n.

Furthermore

$$\int_{\partial\Omega'_n} \frac{h_2(t)}{t - z}\, dt = \int_{\partial\Omega_n} \frac{h_2(t)}{t - z}\, dt - \int_{\partial(\Omega_n \setminus \bar{\Omega}'_n)} \frac{h_2(t)}{t - z}\, dt, \quad z \in \Omega_n.$$

Thus

$$\int_{\partial\Omega'_n} \left|h_2(z)\right|^p |dz| \leqslant M_1 + M_2.$$

That is, by Definition 2.1, $h_2 \in E^p(\Omega')$ for every $0 < p < 1$. This concludes the proof of the lemma. □

Remark 3.2. The above Lemma 3.5 covers the small gap present in the proofs of Theorem 2.2 in [2] and in Theorems 2.1 and 2.8 in [3]. In those papers the authors assumed that for the domains with Ahlfors regular boundaries the two definitions of the Hardy classes were equivalent. That is, while working with the Hardy classes in the sense of Smirnov, they used the definition through the harmonic majorization in order to prove the claim of the above lemma. Otherwise the proofs present in [2,3] are correct.

Finally, we have the following crucial lemma.

Lemma 3.6. *For the cone Δ, let the point w_0 be adjacent to the arc M. Let also $\mathcal{G}(z)$ be the holomorphic function defined by the point-wise summation for every $z \in \Delta$:*

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\phi^{m+1}(z)}.$$

Then

$$\frac{\mathcal{G}^{(k)}(w_0)}{k!} = \sum_{m=0}^{\infty} \sum_{j=0}^{k} \gamma_{m+1,w_0,j} \lambda_{M,m,k-j}.$$

Furthermore

$$\limsup_k \sqrt[k]{\frac{k!}{\mathcal{G}^{(k)}(w_0)}} \geqslant R_{w_0}.$$

Proof. By Lemmas 3.3–3.4 for every $m \in \mathbf{N}$ the functions g_m and $\phi^{-(m+1)}$ are expanded into the power series in the same disk $D(w_0, R_{w_0})$. Therefore for $z \in D(w_0, R_{w_0})$ the following holds:

$$\frac{g_m(z)}{\phi^{m+1}(z)} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \gamma_{m+1,w_0,j} \lambda_{M,m,k-j} \right) (z - w_0)^k. \tag{3.7}$$

On the other hand, the fact that the function $\mathcal{G}(z)$ is analytic in $\Delta \cap D(w_0, R_{w_0})$ implies that for $r > 0$, sufficiently small, one has that

$$\mathcal{G}(z) = \sum_{k=0}^{\infty} \frac{\mathcal{G}^{(k)}(w_0)}{k!} (z - w_0)^k, \quad z \in D(w_0, r).$$

Denote by

$$\rho = \limsup_k \sqrt[k]{\frac{k!}{\mathcal{G}^{(k)}(w_0)}}$$

the radius of convergence of the power series representing the function $\mathcal{G}(z)$ around the point w_0.

By (3.7) one has that

$$\mathcal{G}(w_0) = \sum_{m=0}^{\infty} \frac{g_m(w_0)}{\phi^{m+1}(w_0)} = \sum_{m=0}^{\infty} \beta_{m,0}, \quad \text{where } \beta_{m,0} = \gamma_{m+1,w_0,0} \lambda_{M,m,0}.$$

Again, by (3.7) and the point-wise equality for $z \in D(w_0, \rho)$,

$$\frac{\mathcal{G}(z) - \mathcal{G}(w_0)}{z - w_0} = \frac{1}{z - w_0} \sum_{m=0}^{\infty} \left(\frac{g_m(z)}{\phi^{m+1}(z)} - \beta_{m,0} \right)$$

evaluated at $z = w_0$, one deduces that

$$\mathcal{G}'(w_0) = \sum_{m=0}^{\infty} \beta_{m,1}, \quad \text{where } \beta_{m,1} = \sum_{j=0}^{1} \gamma_{m+1,w_0,j} \lambda_{M,m,1-j}.$$

We repeat the same procedure for every $k = 0, 1, 2, 3, \ldots$, and obtain that

$$\frac{\mathcal{G}^{(k)}}{k!}(w_0) = \sum_{m=0}^{\infty} \beta_{m,k}, \quad \text{where } \beta_{m,k} = \sum_{j=0}^{k} \gamma_{m+1,w_0,j} \lambda_{M,m,k-j}.$$

It follows directly from the above that

$$\frac{\mathcal{G}^{(k)}(w_0)}{k!} = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{k} \gamma_{m+1,w_0,j} \lambda_{M,m,k-j} \right).$$

Or equivalently, in the disk $D(w_0, \rho)$ one can interchange the order of summation just from point-wise convergence:

$$\sum_{m=0}^{\infty} \left(\sum_{k=0}^{\infty} \beta_{m,k} (z - w_0)^k \right) = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} \beta_{m,k} \right) (z - w_0)^k. \tag{3.8}$$

We claim that $\rho = R_{w_0}$.

Actually, for almost all $0 < r < R_{w_0}$, such that $\partial D(w_0, r) \cap M = \{b_{0,r}, b'_{0,r}\}$ is a two point set (Definition 3.1 of the adjacent points guarantee the existence of such r, which actually are sufficiently close to R_{w_0}), the function $\mathcal{G}(z)$ has angular boundary values at the two points $\{b_{0,r}, b'_{0,r}\}$. If

$$F_+(z) = \frac{1}{2\pi i} \int_M \frac{f(\zeta)\, d\zeta}{\zeta - z}, \quad z \in \Delta,$$

the existence of angular boundary values on M for the function $\mathcal{G}(z)$ follows from the identity $\mathcal{G}(z) = f(z) - F_+(z)$, $z \in \Delta$, and the fact that the Cauchy type integrals over Ahlfors regular curves of L^1 functions belong to the Hardy class E^p, $0 < p < 1$ (see the first paragraph of the proof of Lemma 3.5). Denote the angular boundary values of $\mathcal{G}$ at these points by $\mathcal{G}(b_{0,r}), \mathcal{G}(b'_{0,r})$ correspondingly. These coincide with minus the angular boundary value of the exterior Cauchy integral

$$-F_-(w) = -\frac{1}{2\pi i} \int_M \frac{f(\zeta)\, d\zeta}{\zeta - w}, \quad w \in \Delta^c,$$

almost everywhere, since for the boundary values at z_0 of f it holds that $f(z_0) = F_+(z_0) - F_-(z_0)$. (A detailed treatment for the boundary values of Cauchy type integrals along Ahlfors regular curves, can be found in [7] and [10].) That is, $-F_-$ is the pseudo-continuation of $\mathcal{G}$ on the arc $\partial D(w_0, r) \cap \bar{\Delta}^c$. On this arc, and near the points $b_{0,r}, b'_{0,r}$ we take two points $d_{0,r}, d'_{0,r}$ correspondingly. Join the points $b_{0,r}, b'_{0,r}$ by any smooth, nontangent to M, simple curve l lying in Δ and leaving the point w_0 to the left. This curve divides the disk $D(w_0, r)$ into two disjoint simply connected domains Ω_1, Ω_2 having Ahlfors regular boundary and such that $w_0 \in \Omega_1 \subset \Delta$. Then

$$\partial\Omega_1 = \big(\partial D(w_0, r) \cap \bar{\Delta}\big) \cup l, \qquad \partial\Omega_2 = \big(\partial\mathcal{D}(w_0, r) \cap \Delta^c\big) \cup l.$$

Here we would like to note that from the definition of the adjacent points, $\partial\Omega_2 \setminus l \subset \Delta^c$, and hence the part of M with endpoints $b_{0,r}, b'_{0,r}$ is a subset of $\bar{\Omega}_2$. The fact that the function $\mathcal{G}$ has angular boundary values at the points $b_{0,r}, b'_{0,r}$ implies that $\mathcal{G} \in H^\infty(\Omega_1)$. Therefore, for every $k \in \mathbf{N}$ one has that

$$\frac{\mathcal{G}^{(k)}(w_0)}{k!} = \frac{1}{2\pi i} \int\limits_{\partial\Omega_1} \frac{\mathcal{G}(\zeta)\,d\zeta}{(\zeta - w_0)^{k+1}}.$$

Furthermore,

$$\left| \frac{1}{2\pi i} \int\limits_{\partial D(w_0,r)\cap\bar{\Delta}} \frac{\mathcal{G}(\zeta)\,d\zeta}{(\zeta - w_0)^{k+1}} \right| \leqslant \frac{M_r}{r^{k+1}}, \tag{3.9}$$

where $M_r = \max |\mathcal{G}(\zeta)|$, $\zeta \in \partial D(w_0, r) \cap \bar{\Delta}$.

We modify the boundary of the domain Ω_1 by adding to it the arcs $\widehat{b_{0,r}, d_{0,r}}$, $\widehat{b'_{0,r}, d'_{0,r}}$ contained in $\partial D(w_0, r) \cap \Delta^c$. As usual, the orientation of arcs is given by the order of its endpoints. Then for the union of arcs

$$\mathcal{L} = \widehat{d_{0,r}, b_{0,r}} \cup \big(\partial D(w_0, r) \cap \bar{\Delta}\big) \cup \widehat{b'_{0,r}, d'_{0,r}} \cup \widehat{d'_{0,r}, b'_{0,r}} \cup l \cup \widehat{b_{0,r}, d_{0,r}}$$

one has

$$\frac{\mathcal{G}^{(k)}(w_0)}{k!} = \frac{1}{2\pi i} \int\limits_{\partial\Omega_1} \frac{\mathcal{G}(\zeta)\,d\zeta}{(\zeta - w_0)^{k+1}} = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\mathcal{G}(\zeta)\,d\zeta}{(\zeta - w_0)^{k+1}},$$

where for the integration of $\mathcal{G}$ over the part of $\mathcal{L}$ outside the unit disk we use its pseudocontinuation. Thus

$$\left| \frac{1}{2\pi i} \int\limits_{\widehat{d_{0,r},b_{0,r}}\cup(\partial D(w_0,r)\cap\bar{\Delta})\cup\widehat{b'_{0,r},d'_{0,r}}} \frac{\mathcal{G}(\zeta)\,d\zeta}{(\zeta - w_0)^{k+1}} \right| \leqslant \frac{M'_r}{r^{k+1}}, \tag{3.10}$$

where $M'_r = \max |\mathcal{G}(\zeta)|$, $\zeta \in \widehat{d_{0,r}, b_{0,r}} \cup (\partial D(w_0, r) \cap \bar{\Delta}) \cup \widehat{b'_{0,r}, d'_{0,r}}$.

Next, we will show that over the part $\widehat{d_{0,r}, b_{0,r}} \cup l \cup \widehat{b'_{0,r}, d'_{0,r}}$ of $\partial\Omega_2$ one has similar estimate. Actually, for every $m \in \mathbf{N}$ the function

$$\frac{g_m(\zeta)}{\phi^{m+1}(\zeta)} \in \mathcal{H}(\bar{\Omega}_2),$$

since it is analytic in the disk $D(w_0, R_{w_0})$ containing properly the closure of Ω_2. Therefore for every $k = 0, 1, 2, \ldots,$

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega_2} \frac{\frac{g_m(\zeta)}{\phi^{m+1}(\zeta)}}{(\zeta - w_0)^k}\,d\zeta = 0, \quad \text{for all } m = 0, 1, 2, \ldots.$$

Hence for every $m = 0, 1, 2, \ldots$ and for every $k = 0, 1, 2, \ldots$ the following holds:

$$\frac{1}{2\pi i} \int_{\widehat{d_{0,r}, b_{0,r}} \cup l \cup \widehat{b'_{0,r}, d'_{0,r}}} \frac{\frac{g_m(\zeta)}{\phi^{m+1}(\zeta)}}{(\zeta - w_0)^k} \, d\zeta = -\frac{1}{2\pi i} \int_{\widehat{d'_{0,r}, d_{0,r}}} \frac{\frac{g_m(\zeta)}{\phi^{m+1}(\zeta)}}{(\zeta - w_0)^k} \, d\zeta. \tag{3.11}$$

Furthermore, the definition of the quenching function ϕ in (1.1) and the construction of the exhaustion of Δ (see (2.1)), imply that for every $\zeta' \in M$ one has that $|\phi(\zeta')| \leqslant e^{x_M^n}$. On the other hand for the points $d'_{0,r}, d_{0,r}$ of the circle $\partial D(w_0, r)$ which are sufficiently close to the point $\{x \in \mathbf{R}\colon\ x > x_M\} \cap \partial D(w_0, r)$ one has that for every $\zeta \in \widehat{d'_{0,r}, d_{0,r}}$ the inequality $|\phi(\zeta)| > e^{x_M^n + \epsilon}$ holds for some $\epsilon > 0$. This is possible because the adjacent points are close to x_M from below (see Remark 3.1). Thus, if $\zeta' \in M$ and $\zeta \in \widehat{d'_{0,r}, d_{0,r}}$ then

$$\frac{|\phi(\zeta)|}{|\phi(\zeta')|} \geqslant C_M > 1, \tag{3.12}$$

where C_M is a constant. Thus for $\zeta \in \widehat{d'_{0,r}, d_{0,r}}$,

$$\left| \frac{g_m(\zeta)}{\phi^m(\zeta)} \right| \leqslant A'_{\max} C_M^{-m} \int_M \bigl| f(y) \bigr| \, dy,$$

where

$$\left| \frac{\phi(\zeta) - \phi(y)}{\zeta - y} \right| \leqslant A'_{\max} = \frac{2C_{\max}}{\operatorname{dist}(\widehat{d'_{0,r}, d_{0,r}}, M)}, \quad \text{where } C_{\max} = \max_{\zeta \in \widehat{d'_{0,r}, d_{0,r}}} \bigl| \phi(\zeta) \bigr|.$$

This implies that

$$\sum_{m=0}^{\infty} \int_{\widehat{d'_{0,r}, d_{0,r}}} \frac{\left| \frac{g_m(\zeta)}{\phi^{m+1}(\zeta)} \right|}{|(\zeta - w_0)^{k+1}|} \, d\zeta < \infty.$$

Thus for almost every $\zeta \in \widehat{d'_{0,r}, d_{0,r}}$ and for every $k \in \mathbf{N}$ the sum

$$\Psi_k(\zeta) = \sum_{m=0}^{\infty} \frac{\frac{g_m(\zeta)}{\phi^{m+1}(\zeta)}}{(\zeta - w_0)^{k+1}}$$

is well-defined continuous, bounded function and

$$\int_{\widehat{d'_{0,r}, d_{0,r}}} \Psi_k(w) \, dw = \sum_{m=0}^{\infty} \int_{\widehat{d'_{0,r}, d_{0,r}}} \frac{\frac{g_m(w)}{\phi^{m+1}(w)}}{(w - w_0)^{k+1}} \, dw.$$

In order to interchange the order of integration and summation in the sum of the left-hand side of (3.11) over l one uses Egoroff theorem. Over the rest of the curve the integral of

$-F_-$, by changing the contour of integration, is equal to the integral of the $\sum_m \frac{g_m(z)}{\phi^{m+1}(z)}$. Thus, abusing slightly the notation, one has

$$\left|\sum_{m=0}^{\infty} \int\limits_{\widehat{d_{0,r},b_{0,r}} \cup l \cup \widehat{b'_{0,r},d'_{0,r}}} \frac{\frac{g_m(w)}{\phi^{m+1}(w)}}{(w-w_0)^{k+1}}\,dw\right| = \left|\int\limits_{\widehat{d'_{0,r},d_{0,r}}} \Psi_k(w)\,dw\right| \leqslant \frac{c}{r^{k+1}}, \tag{3.13}$$

for some constant $c > 0$. Combining (3.10), (3.13) we obtain

$$\left|\frac{\mathcal{G}(k)}{k!}\right| \leqslant \frac{c'}{r^{k+1}},$$

where $c' > 0$ is a suitable constant. It follows that the corresponding power series of $\mathcal{G}(z)$ about w_0 has radius of convergence at least r. But $r \to R_{w_0}$ from below, thus we get that the radius of convergence for the power series of $\mathcal{G}(z)$ about w_0 is at the least equal to R_{w_0}. Summarizing, it implies that for $\beta_i = \sum_{m=0}^{\infty} \beta_{m,i}$ one has

$$\begin{aligned}
\sum_{i=0}^{\infty} \beta_i (z-w_0)^i &= \sum_{i=0}^{\infty}\sum_{m=0}^{\infty} \beta_{m,i}(z-w_0)^i \\
&= \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} \gamma_{m+1,w_0,j}\lambda_{M,m,i-j}\right)(z-w_0)^i \\
&= \lim_{m\to\infty} \frac{1}{2\pi i}\int\limits_M f(\zeta)\frac{\phi^m(\zeta)}{\phi^{m+1}(z)}\frac{\phi(\zeta)-\phi(z)}{\zeta - z}\,d\zeta,
\end{aligned}$$

whenever $z \in D(w_0, R_{w_0}) \cap \Delta$. $\square$

The last lemma implies that minus the exterior Cauchy type integral which is equal to

$$-F_-(z) = -\frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)}{\zeta - z}\,d\zeta, \tag{3.14}$$

whenever $z \notin \bar{\Delta}$, can be continued analytically through the open arc M. This follows from the fact that $\mathcal{G}(z) = -F_-(z)$ almost everywhere on M and that $\mathcal{G}(z)$ has analytic continuation across the arc M. This means for the radius $R_{w_0} = |z_0^1 - w_0|$ the function $\mathcal{G}$ can be expanded into the power series in the disk $D(w_0, R_{w_0})$, that is

$$\mathcal{G}(z) = \sum_{i=0}^{\infty} \mu_i (z-w_0)^i, \quad z \in D(w_0, R_{w_0}),\ \mu_i = \frac{\mathcal{G}^{(i)}(w_0)}{i!}. \tag{3.15}$$

At this point we conclude the proof of the second part of Theorem 2.1.

Proof. It is enough to show that for every $z \in \Delta$ one has that

$$f(z) = \frac{1}{2\pi i}\int\limits_M \frac{f(\zeta)\,d\zeta}{\zeta - z} + g(z), \tag{3.16}$$

where $g(z)$ is holomorphic function which has analytic continuation across M. That is, there exists open, bounded, connected set U containing M and satisfying $\partial U \cap \bar{M} = \{z_1^0, z_2^0\}$, and $\hat{g} \in \mathcal{H}(\Delta \cup U)$ such that $\hat{g}(z) = g(z)$, $z \in \Delta$. To be more precise, Lemma 3.5 implies that $F_+(z) = \frac{1}{2\pi i} \int_M \frac{f(\zeta)\, d\zeta}{\zeta - z}$ belongs to $E^p(\Delta_\tau)$ for every $p < 1$ and obviously the function $g(z)$ is holomorphic in $\bar{\Delta}_\tau$. These two facts, the observation that $f \in L^1(\partial \Delta_\tau)$ for almost all $\tau > 0$ and two-fold application of the Smirnov theorem is the essence of the proof of Theorem 2.8 in [3] (even though it was proven there for different type of simply connected domains). Thus the conclusion of Theorem 2.8 from [3] is valid in this case also. Hence (3.16) implies the second part of Theorem 2.1.

Actually, Lemma 3.1 and relation (3.2) imply that in order to prove the last part one needs to show that the holomorphic function, defined as a point-wise limit,

$$\mathcal{G}(z) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\phi^{m+1}(z)} = f(z) - \frac{1}{2\pi i} \int_M \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Delta,$$

can be continued analytically across the open arc M. Lemmas 3.2–3.4 show that every term $\frac{g_m(z)}{\phi^{m+1}(z)}$ of the above series is expanded into a power series about an adjacent point w_0 in a disk of common radius $D(w_0, R_{w_0})$. Furthermore, R_{w_0} is smaller than the radius of convergence for every m. Lemma 3.6 shows that in the disk $D(w_0, R_{w_0})$ the sum of the power series that represent the terms $\frac{g_m(z)}{\phi^{m+1}(z)}$ is equal to the power series, whose coefficients are the sums of coefficients of the same order, in the same disk. That is the changing of summation order is established by Lemma 3.6.

Therefore, for $z \in \Delta$,

$$f(z) = F_+(z) + \mathcal{G}(z) = \frac{1}{2\pi i} \int_M \frac{f(\zeta)\, d\zeta}{\zeta - z} + \mathcal{G}(z),$$

where the holomorphic function $\mathcal{G}(z)$ has analytic continuation across the arc M. From the discussion above one concludes the proof of Theorem 2.1. □

Example 3.1. The following example shows that $\mathcal{NH}_M^1(\Delta) \neq E^1(\Delta)$. If $d \in \partial \Delta$ is such that $d^{-1} \in \partial \Delta \setminus \bar{M}$ then the function $f(z) = \sum_{i=0}^{\infty} (dz)^i = \frac{1}{1 - dz}$, $z \in \Delta$, belongs to the class of holomorphic functions representable by Carleman formula (1.4) but does not belong to the Hardy space $E^1(\Delta)$.

4. An extension theorem

In this section the cone Δ and its base M are as in the previous sections. For the quenching function ϕ defined by (1.1) we consider the sequence of functions

$$g_m(z) = \int_M f(\zeta) \phi^m(\zeta) \frac{\phi(z) - \phi(\zeta)}{z - \zeta}\, d\zeta, \quad m = 0, 1, 2, 3, \ldots. \tag{4.1}$$

From the definition of the sequence $\{g_m(z)\}_m$ follows that these functions are holomorphic everywhere in $\mathbf{C} \setminus M$. One can show that $g_m(z)$ is analytic in $D(w_0, R_{w_0})$ for every point

$w_0 \in \Delta$ adjacent to the base M (see Lemma 3.4). The size of the disk $D(w_0, R_{w_0})$ does not depend on m and the radius is equal to $R_{w_0} = |w_0 - z_0^1|$. Therefore, for every m one has that

$$\frac{g_m(z)}{\phi^{m+1}(z)} = \sum_{l=0}^{\infty} \beta_{m,w_0,l}(z - w_0)^l, \quad z \in D(w_0, R_{w_0}), \tag{4.2}$$

$$\Bigl(\limsup_l \sqrt[l]{|\beta_{m,w_0,l}|}\Bigr)^{-1} \geqslant R_{w_0}. \tag{4.3}$$

Next, for every $l \in \mathbf{N}$ we define the sequence of coefficients

$$\beta_l = \sum_{m=0}^{\infty} \beta_{m,w_0,l}. \tag{4.4}$$

Now we are ready to formulate an extension theorem. The proof of this theorem goes along the steps of the proof of Theorem 2.1 (the second part), but in reverse order and therefore is omitted.

Theorem 4.1. *Let M be as before and $f \in L^1(M)$. If for some adjacent point $w_0 \in \Delta$,*

$$\Bigl(\limsup_l \sqrt[l]{|\beta_l|}\Bigr)^{-1} \geqslant R_{w_0}, \tag{4.5}$$

and the equality

$$\sum_{m=0}^{\infty} \sum_{l=1}^{\infty} \beta_{m,l}(z - w_0)^l = \sum_{l=1}^{\infty} \beta_l (z - w_0)^l \tag{4.6}$$

holds for $z \in \Delta \cap D(w_0, R_{w_0})$, then f extends to a function Φ holomorphic in Δ belonging to the class $\mathcal{NH}_M^1(\Delta)$. Furthermore this extension is defined by

$$\Phi(z) = \lim_{m\to\infty} \frac{1}{2\pi i} \int_M \frac{\phi^m(\zeta)}{\phi^m(z)} \frac{f(\zeta)\, d\zeta}{\zeta - z}, \tag{4.7}$$

whenever $z \in \Delta$. The convergence is uniform over the compact subsets of Δ.

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